Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
Spring 2005

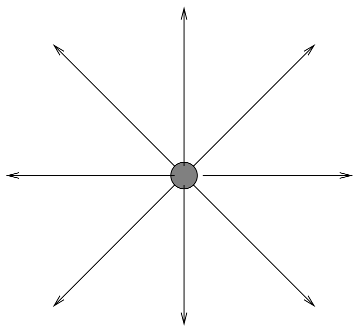
Solution to øving 3

Guidance: January 27 and 28

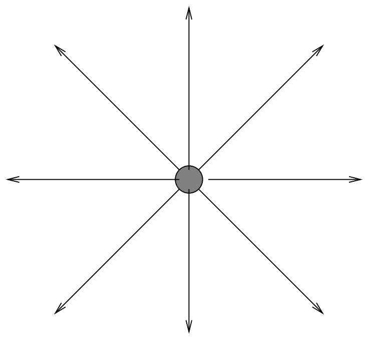
Exercise 1

a)

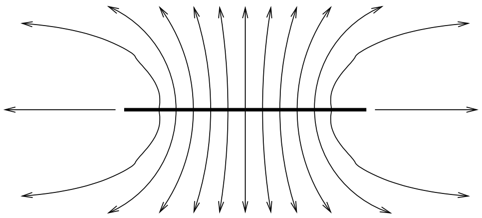
stav, plan normalt på, nært



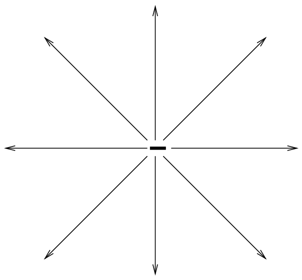
stav, plan normalt på, langt unna



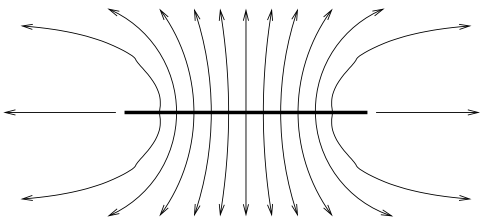
stav, plan inneholder staven, nært



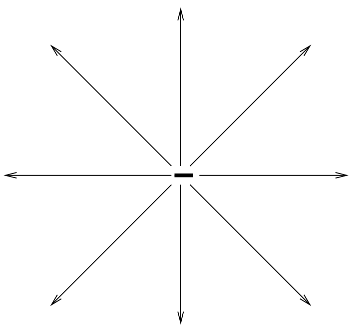
stav, plan inneholder staven, langt unna



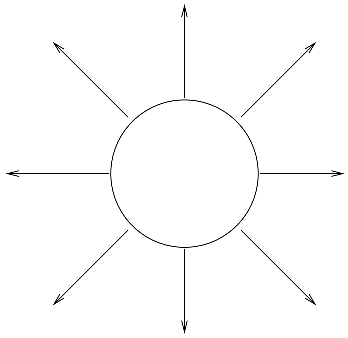
skive, plan normalt på, nært



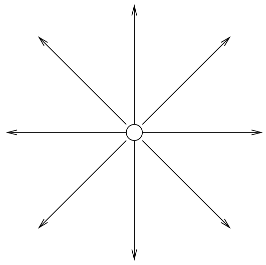
skive, plan normalt på, langt unna



skive, plan inneholder skiva, nært

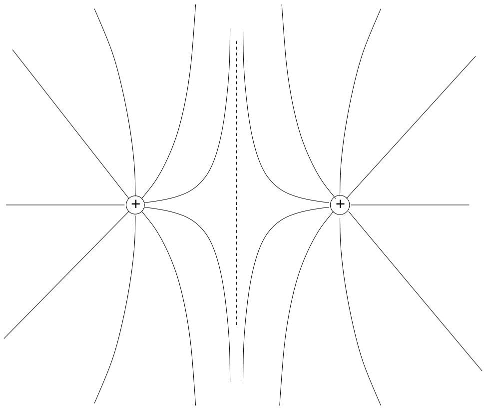


skive, plan inneholder skiva, langt unna

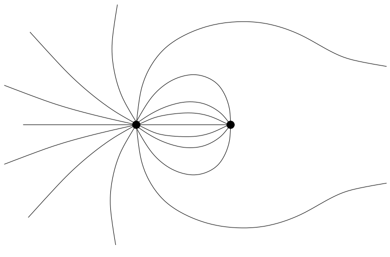


Comment: These figures are only *qualitative*, not *quantitative*. Note that far away (i.e., the four figures in the right column), everything looks like a point charge. Closer to the charge distribution, one can usually apply symmetry arguments combined with what one knows about the electric field in the vicinity of point charges, to sketch a reasonable picture of the field lines.

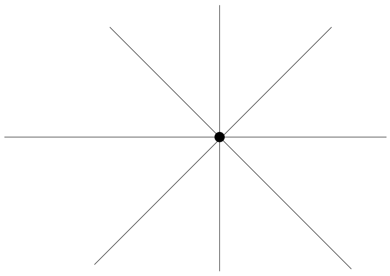
b) (i) Field lines around two equal positive point charges:



(ii) Field lines around two point charges $-2q$ og q:
"Closeup" (equally many field lines out pr positive charge q as in pr negative charge $-q$, therefore twice as many field lines in towards $-2q$ as out from q. The remaining field lines must come from infinity):

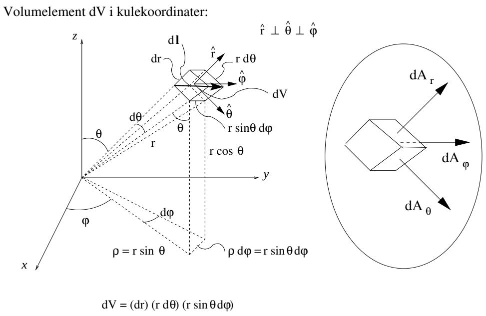


Very far away from the charges (now we see essentially a point charge $-2q + q = -q$, i.e., the field lines are directed in towards the charge):



Exercise 2

a) We take the hint given in the text and start with the following figure:



In the figure, we have drawn a line element $d\boldsymbol{l}$, which in spherical coordinates, in its most general form, consists of a displacement along the three orthogonal directions specified by the above mentioned unit vectors. We observe that such a displacement, from the point (r, θ, ϕ) to the point $(r + dr, \theta + d\theta, \phi + d\phi)$, corresponds to the vector $d\boldsymbol{l}$ diagonally through the volume element dV. We see from the figure that this vector has components dr along $\hat{r}$, $r\ d\theta$ along $\hat{\theta}$ and $r \sin\theta\ d\phi$ along $\hat{\phi}$. Thus:

$$d\boldsymbol{l} = dr\ \hat{r} + r\ d\theta\ \hat{\theta} + r \sin\theta\ d\phi\ \hat{\phi}$$

Note that while in cartesian coordinates, the components of the vector $d\boldsymbol{l}$ are always the same (dx, dy, dz), whereas in spherical coordinates, two of them depend upon "where we are": The component along $\hat{\theta}$ is proportional to r, i.e., the distance from the origin, while the component along $\hat{\phi}$ also depends upon the angle θ (i.e., the "longitude", if we imagine the z-axis through the poles and equator in the xy-plane). For example, $dl_\phi = r \sin 0 = 0$ if we start in $\theta = 0$. Not unreasonable: If we stand on one of the poles, a small step will always be in the south (or north) direction, never east or west. And if we are standing on the equator, i.e., in $\theta = \pi/2$, we obtain $dl_\phi = r \sin \pi/2\ d\phi = r\ d\phi$. Also not unreasonable: Here, east, west, south and north are directions "on an equal footing", so that $dl_\theta = r\ d\theta$ and $dl_\phi = r\ d\phi$ should be expressed "in the same form".

From the figure, we easily find the three surface elements with unit normals along $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$, respectively:

$$\begin{aligned}
d\boldsymbol{A}_r &= (r\,d\theta)(r\sin\theta\,d\phi)\hat{r} \\
&= r^2\sin\theta\,d\theta\,d\phi\,\hat{r} \\
d\boldsymbol{A}_\theta &= (dr)(r\sin\theta\,d\phi)\hat{\theta} \\
&= r\,dr\sin\theta\,d\phi\,\hat{\theta} \\
d\boldsymbol{A}_\phi &= (dr)(r\,d\theta)\hat{\phi} \\
&= r\,dr\,d\theta\,\hat{\phi}
\end{aligned}$$

Note that these three quantities are *vectors*, with absolute value equal to the area of the surface element (e.g. dA_r) and direction normal to the surface (e.g. $\hat{r}$). We need both the *size* and the *orientation* in order to have a precise description of a surface!

Finally, we see from the figure that the volume of the volume element becomes

$$dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\,dr\sin\theta\,d\theta\,d\phi$$

b) Now, we can determine the volume of a sphere with radius R by integrating the volume element dV over all values of θ and ϕ, and r from 0 to R:

$$\begin{aligned}
V(R) &= \int_{r<R} dV = \int_0^R r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi \\
&= \frac{1}{3}R^3\cdot 2\cdot 2\pi = \frac{4\pi}{3}R^3
\end{aligned}$$

Note that when we integrate over ϕ from 0 to 2π, we must integrate over θ from 0 to π, and not 2π, in order to cover all the solid angles (i.e., all directions) *once*, and not twice.

The surface area of a sphere with radius R can be found by integrating the surface element dA_r (i.e., the absolute value of $d\boldsymbol{A}_r$) over all values of θ and ϕ, keeping $r = R$ fixed:

$$A(R) = \int_{r=R} dA_r = R^2 \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi = R^2\cdot 2\cdot 2\pi = 4\pi R^2$$

c) The given charge density is positive (or zero) everywhere inside the sphere. It grows linearly with the distance from the centre of the sphere. Furthermore, the term $\cos^2\theta$ yields the highest charge density on the two "poles" (i.e., $\theta = 0$ or $\theta = \pi$) and the smallest charge density (zero) in the equatorial plane (i.e., $\theta = \pi/2$).
A small volume element dV of the sphere contains a charge

$$dq = \rho\,dV$$

The total charge of the sphere is obtained by integrating dq over the volume of the sphere. We use dV as given in *a*) and obtain:

$$
\begin{aligned}
Q &= \int dq \\
&= \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_0 \frac{r}{R} \cos^2\theta r^2 \; dr \sin\theta \; d\theta \; d\phi \\
&= \rho_0 \left(\int_{r=0}^{R} \frac{r^3}{R} \; dr \right) \left(\int_{\theta=0}^{\pi} \cos^2\theta \sin\theta \; d\theta \right) \left(\int_{\phi=0}^{2\pi} d\phi \right) \\
&= \rho_0 \; |_0^R \; \frac{r^4}{4R} \; |_0^\pi \; \left(-\frac{1}{3} \cos^3\theta \right) \; |_0^{2\pi} \; \phi \\
&= \rho_0 \frac{R^3}{4} \cdot \frac{2}{3} \cdot 2\pi \\
&= \frac{\rho_0 \pi R^3}{3}
\end{aligned}
$$

Have we done the calculation correctly? Well, at least we have the correct dimension: A charge pr unit volume, ρ_0, multiplied with R^3, which is a volume.

In other words: Nothing mysterious about such multiple integrals. You simply integrate each of the variables separately. In our examples, the integrand was always independent of the angle ϕ, so the integral over that variable simply gave a factor of 2π. Further, the θ dependence of the charge density in the final example was carefully chosen, so that the integral over θ became an easily tractable one.

Also note that usually, we don't bother to write explicity $\int\int\int dV$, but simply $\int dV$, even though there are actually three integrals involved. It will always be clear in a given problem whether we are supposed to integrate over a line, a surface, or a volume.