Institutt for fysikk, NTNU
TFY4155/FY1303 Elektrisitet og magnetisme
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Solution to øving 4

Guidance Monday February 2

Exercise 1

The potential difference ΔV between two points in space is given by

$$\Delta V = V_B - V_A = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l}$$

In this exercise, we have a uniform electric field $\boldsymbol{E} = E_0\, \hat{x}$, so we may write

$$\Delta V = -E_0\, \hat{x} \cdot \int_A^B d\boldsymbol{l}$$

where A represents the origin, $(0,0)$, and B the three points given in the text. We find:

(i)

$$\int_A^B d\boldsymbol{l} = \int_{(0,0)}^{(a,0)} d\boldsymbol{l} = a\,\hat{x}$$

hence

$$\Delta V = -E_0\, \hat{x} \cdot a\,\hat{x} = -E_0 a$$

(ii)

$$\int_A^B d\boldsymbol{l} = \int_{(0,0)}^{(0,a)} d\boldsymbol{l} = a\,\hat{y}$$

hence

$$\Delta V = -E_0\, \hat{x} \cdot a\,\hat{y} = 0$$

(iii)

$$\int_A^B d\boldsymbol{l} = \int_{(0,0)}^{(a,a)} d\boldsymbol{l} = a\,\hat{x} + a\,\hat{y}$$

hence

$$\Delta V = -E_0\, \hat{x} \cdot (a\,\hat{x} + a\,\hat{y}) = -E_0 a$$

Exercise 2

a) With our choice for the polar angle θ, we see from the figure that

$$\begin{aligned} x &= r\sin\theta \\ z &= r\cos\theta \\ r &= \sqrt{x^2+z^2} \end{aligned}$$

b) We use the super position principle to determine the potential from the two point charges. With the point (x, z) in a distance r_1 from q and r_2 from $-q$, we have

$$\begin{aligned} V(x,z) &= \frac{q}{4\pi\varepsilon_0 r_1} - \frac{q}{4\pi\varepsilon_0 r_2} \\ &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+(z-a/2)^2}} - \frac{1}{\sqrt{x^2+(z+a/2)^2}}\right) \end{aligned}$$

The distances r_1 and r_2 in terms of x and z are found directly by looking at the figure.

The potential on the x axis is

$$V(x,0) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+a^2/4}} - \frac{1}{\sqrt{x^2+a^2/4}}\right) = 0$$

The potential on the z axis is

$$V(0,z) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|}\right)$$

Note that we must take absolute values in order to have *one* expression valid *throughout* the z axis. With $z > a/2$:

$$\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|} = \frac{1}{z-a/2} - \frac{1}{z+a/2} = \frac{a}{z^2-a^2/4}$$

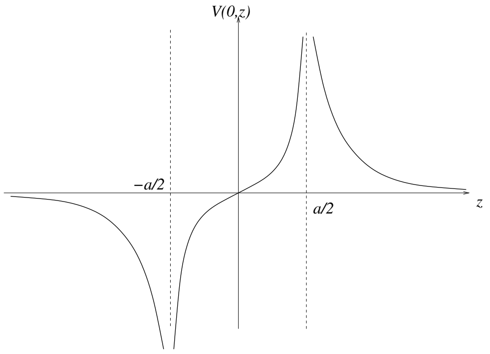
With $z < -a/2$:

$$\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|} = -\frac{1}{z-a/2} + \frac{1}{z+a/2} = -\frac{a}{z^2-a^2/4}$$

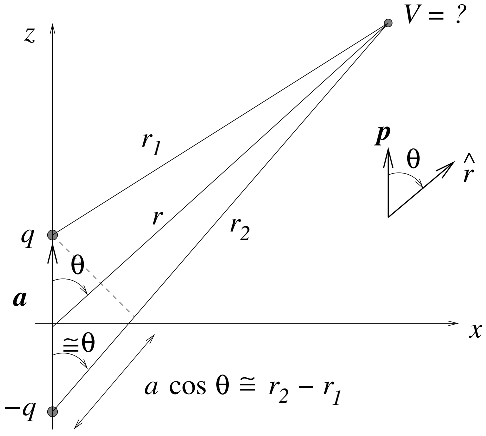
With $-a/2 < z < a/2$:

$$\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|} = -\frac{1}{z-a/2} - \frac{1}{z+a/2} = -\frac{2z}{z^2-a^2/4} = \frac{2z}{a^2/4-z^2}$$

A sketch of $V(0,z)$:



c) We use the hint given in the text, in addition to the following figure, and obtain:



$$\begin{aligned}
V(r,\theta) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \\
&= \frac{q}{4\pi\varepsilon_0}\cdot\frac{r_2-r_1}{r_1 r_2} \\
&\simeq \frac{q}{4\pi\varepsilon_0}\cdot\frac{a\cos\theta}{r^2} \\
&= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \\
&= \frac{pr\cos\theta}{4\pi\varepsilon_0 r^3} \\
&= \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3}
\end{aligned}$$

Alternatively, proceeding a bit more slowly: From the figure, we see that

$$\begin{aligned}
r_1 &\simeq r - \frac{a}{2}\cos\theta \\
r_2 &\simeq r + \frac{a}{2}\cos\theta
\end{aligned}$$

When $r \gg a$ we may expand both $1/r_1$ and $1/r_2$ in power series around $1/r$:

$$\begin{aligned}
\frac{1}{r_1}-\frac{1}{r_2} &\simeq \frac{1}{r-\frac{a}{2}\cos\theta}-\frac{1}{r+\frac{a}{2}\cos\theta} \\
&= \frac{1}{r}\left[\left(1-\frac{a\cos\theta}{2r}\right)^{-1}-\left(1+\frac{a\cos\theta}{2r}\right)^{-1}\right] \\
&\simeq \frac{1}{r}\left[1+\frac{a\cos\theta}{2r}-1+\frac{a\cos\theta}{2r}\right] \\
&= \frac{a\cos\theta}{r^2}
\end{aligned}$$

Is it *reasonable* that the potential from an electric dipole falls off faster than the potential from a point charge (i.e., an electric "monopole")? Yes, because the negative and the positive point charges of the dipole contribute with opposite signs to the total potential. Thus, the contributions to the potential from the two point charges partly cancel each other. (On the x axis, the two contributions cancel *exactly.*)

Extra, if you wonder how one should proceed in order to find the dominating *correction* to the result obtained above:
A first thought might be to continue the series expansion that was started above, and include sufficiently many terms so that a dominating correction was obtained. If we include one more term, nothing new is obtained since that term comes with the same sign in the expansion of the two square roots, and therefore (with the minus sign in front of one of them) cancel each other. We must include *two* more terms:

$$
\begin{aligned}
&\frac{1}{r}\left[\left(1-\frac{a\cos\theta}{2r}\right)^{-1}-\left(1+\frac{a\cos\theta}{2r}\right)^{-1}\right] \\
= \; &\frac{1}{r}\left[1+\frac{a\cos\theta}{2r}+\left(\frac{a\cos\theta}{2r}\right)^2+\left(\frac{a\cos\theta}{2r}\right)^3+\ldots-\left(1-\frac{a\cos\theta}{2r}+\left(\frac{a\cos\theta}{2r}\right)^2-\left(\frac{a\cos\theta}{2r}\right)^3+\ldots\right)\right] \\
= \; &\frac{a\cos\theta}{r^2}-\frac{a^3\cos^3\theta}{4r^4}+\ldots \\
= \; &\frac{a\cos\theta}{r^2}\left(1-\frac{a^2\cos^2\theta}{4r^2}+\ldots\right)
\end{aligned}
$$

Here, we used the series expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ (valid for $|x| < 1$). Not a bad try. However, there is a catch here: The *starting point* for this series expansion was an approximation itself, namely

$$
\begin{aligned}
r_1 &\simeq r - \frac{a}{2}\cos\theta \\
r_2 &\simeq r + \frac{a}{2}\cos\theta
\end{aligned}
$$

And the errors we do in these approximations are of the same order of magnitude as the correction term we are looking for!
The solution is obtained by going back to the exact expression for V, with r_1 and r_2 expressed in terms of cartesian coordinates x and z. The calculation is not difficult, but rather tedious, so the details are skipped here. If my calculation is correct, the answer is

$$
\frac{1}{r_1}-\frac{1}{r_2}=\frac{a\cos\theta}{r^2}\left[1-\frac{3a^2}{8r^2}\left(1-\frac{5}{3}\cos^2\theta\right)+\ldots\right]
$$

Here, we have included all corrections that are an order a^2/r^2 smaller than the dominating result. The next term in this series will be further reduced, by an additional factor a^2/r^2. In series expansions like this, the first term that is *not* included will always be smaller than the last term that we did include, since the series is a polynomial with increasing powers of a parameter which is small compared to 1. (In our particular case, we see that for directions given by $\cos^2\theta \simeq 3/5$, the first correction actually disappears. In that case, one may have to look at the next term in the series expansion.)

Exercise 3

a) **C**

$$\boldsymbol{F} = q\boldsymbol{E} = m\boldsymbol{a}$$

Newton's 2. law! Here, $q = -e$, so the acceleration of the electron becomes

$$\boldsymbol{a} = -\frac{e}{m}\boldsymbol{E}$$

i.e., to the left.

b) **C**

$$\Delta V = -\int \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

provided that

$$d\boldsymbol{l} \perp \boldsymbol{E}$$

c) **D**
A is nonsense, V is a *scalar*, and it is E, not V, that is measured in units of N/C.

d) **A**

$$\begin{aligned} U &= \sum_{i<j} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}} \\ &= \frac{1}{4\pi\varepsilon_0 l}(q_1 q_2 + q_1 q_3 + q_2 q_3) \\ &= 9 \cdot 10^9 \cdot \frac{1}{0.5}(50 + 75 + 150) \cdot 10^{-6-6} \\ &= 18 \cdot 10^9 \cdot 275 \cdot 10^{-12} \\ &= 4.95 \text{ J} \end{aligned}$$

e) **C**
Like the previous one, but with $q_2 = -10\ \mu\text{C} \Rightarrow$

$$U = 18 \cdot 10^9 \cdot (-50 + 75 - 150) \cdot 10^{-12} = -2.25 \text{ J}$$

f) **B**

$$U = \frac{e^2}{4\pi\varepsilon_0 r} = e \cdot \frac{e}{4\pi\varepsilon_0 r} = e \cdot 1.6 \cdot 10^{-19} \cdot 9 \cdot 10^9 \cdot 10^{10} = 14.4 \text{ eV}$$

g) **D**

Energy conservation yields

$$\frac{1}{2}mv^2 = qV$$

I.e., acceleration of a particle with charge q and mass m through a potensial difference V results in a reduction of potential energy, qV, and a corresponding increase in kinetic energy, $mv^2/2$. Equal speed for the two particles then implies

$$\frac{q_\alpha V_\alpha}{m_\alpha} = \frac{q_{\text{Be}} V_{\text{Be}}}{m_{\text{Be}}}$$

in other words

$$\frac{V_{\text{Be}}}{V_\alpha} = \frac{q_\alpha m_{\text{Be}}}{q_{\text{Be}} m_\alpha} = \frac{2 \cdot 9}{4 \cdot 4} = \frac{9}{8}$$