Institutt for fysikk, NTNU
TFY4155/FY1003: Elektrisitet og magnetisme
Vår 2005

Øving 6

Guidance: February 17 and 18
To be delivered by: Monday February 21

Exercise 1

The largest electric field that can be maintained in air is about 3 MV/m. Larger fields result in a discharge (socalled corona discharge). In the lectures, we have shown that a metal sphere will have all its net charge on the surface. We have also shown that the electric field at the surface is $E = \sigma/\varepsilon_0$, where σ is the surface charge density.

a) What is then the maximum surface charge density that a metal surface can maintain without resulting in a discharge?

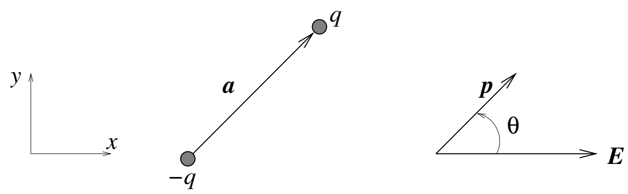
b) What is the minimum radius of a metal sphere if it is supposed to maintain a charge 1C? [Correct answer is either 25 nm, 1.5 mm, 6.6 cm, 54 m or 2.3 km]

c) A typical metal consists of atoms arranged in a crystal structure, with a nearest neighbour distance of about 0.3 nm. What is then the number of surface atoms pr m^2? You may assume that the surface atoms are arranged in a regular *quadratic* lattice. [Extra: Will your answer be the same if the surface atoms are arranged in a regular *triangular* lattice?]

d) The surface charge in *a*) is located on the metal described in *c*). Assuming all the net charge is distributed only in the outermost atomic layer at the surface, how big fraction of the atoms in this layer has got an extra electron?
[Correct answer is either $3.3 \cdot 10^{-9}$, $1.5 \cdot 10^{-5}$, $4.3 \cdot 10^{-3}$ or 0.17]

Exercise 2 ($\simeq$ exercise 3, exam 15. August 2003)

An electric dipole consists of two point charges q and $-q$ with a (fixed) distance a between them. The dipole is placed in a homogeneous "external" electrostatic field $\boldsymbol{E} = E\hat{x}$. Assume that the dipole is located in the xy plane, and in such a way that the vector $\boldsymbol{a}$ from $-q$ to q, and therefore also the dipole moment $\boldsymbol{p} = q\boldsymbol{a}$, makes an angle θ with $\boldsymbol{E}$. The angle θ is measured *counterclockwise* from the x axis, as shown in the figure.



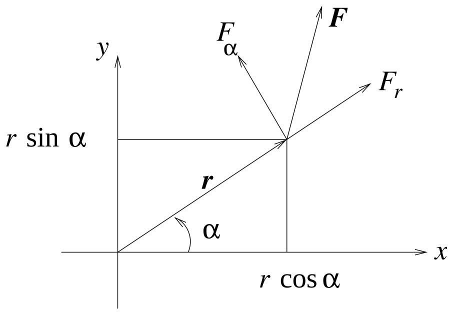
a) What is the total force (from the external field $\boldsymbol{E}$) on the dipole?

b) From the mechanics course, we remember that the *torque* $\boldsymbol{\tau}$ around a given axis is defined as $\boldsymbol{\tau} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i$, where $\boldsymbol{r}_i$ is the "arm" from the axis to the position where the force $\boldsymbol{F}_i$ acts. (You will find some information about the cross product at the end of this exercise.)
Show that for the electric dipole in the homogeneous field, the torque around the axis that is perpendicular to the plane containing $\boldsymbol{p}$ and $\boldsymbol{E}$, and going through the center of the dipole, is

$$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E} = -\boldsymbol{E} \times \boldsymbol{p} = -pE \sin\theta \, \hat{z}$$

c) Finally, you are supposed to find an expression for the potential energy $U(\theta)$ of the electric dipole described above. Sketch $U(\theta)$. Which orientation of the dipole (relative to $\boldsymbol{E}$) represents a stable equilibrium?

Some helpful information concerning c):
Let us for simplicity stay within the xy plane only. A force $\boldsymbol{F} = F_x \, \hat{x} + F_y \, \hat{y} = F_r \, \hat{r} + F_\alpha \, \hat{\alpha}$ acting in a position $\boldsymbol{r} = r\cos\alpha \, \hat{x} + r\sin\alpha \, \hat{y}$ then yields a torque $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ around the z axis:



Further, we know that the force $\boldsymbol{F}$ may be derived from the potential energy U using the gradient operator: $\boldsymbol{F} = -\nabla U$. In polar coordinates (r, α), we have

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\alpha}\frac{1}{r}\frac{\partial}{\partial \alpha}$$

With this information, one can show that

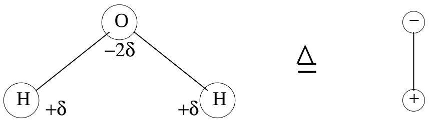
$$\tau = -\frac{\partial U}{\partial \alpha},$$

and hence

$$dU = -\tau \, d\alpha$$

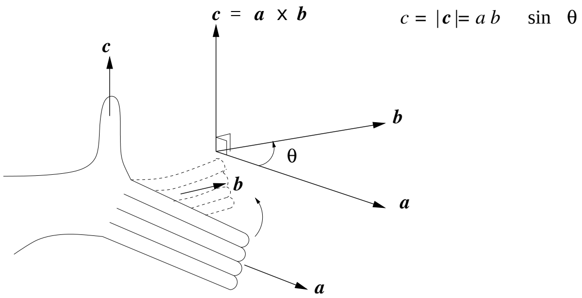
since U does not depend on r in our case. (We have a fixed value $r = a/2$ for the dipole.)

Comments: An electric insulator, i.e., a *dielectric*, typically consists of molecules with zero net charge. However, the internal charge distribution (i.e., the positions of the atomic nuclei and the electrons) may be such that the molecule has a nonzero dipole moment. Put another way: The "center of charge" for the positive charges of the molecule (i.e., the atomic nuclei) is not in the same position as the center of charge for the negative charges of the molecule (i.e., the electrons). Such *polar* molecules may be described in terms of simple electric dipoles, like the one we have discussed above. (Well, actually they *are* electric dipoles.) An example: Water, H_2O.



Since oxygen is more electronegative than hydrogen (i.e., it "wants" an extra electron more than hydrogen does), the electron distribution will be somewhat distorted in the direction of the oxygen atom within a water molecule. This means that close to the O atom, we have a small net negative charge, e.g. -2δ. Since the molecule is overall electrically neutral (and given the symmetry of the water molecule), there must be a small positive net charge $+\delta$ near each H atom.
A dielectric may also consist of atoms or molecules *without* such a polar charge distribution, i.e., with zero (permanent) electric dipole moment. However, if such a material is placed in an external electric field, the electrons and the nuclei will be pulled in separate directions, resulting in an *induced* electric dipole moment $\boldsymbol{p}_{\text{ind}}$ directed along $\boldsymbol{E}$. The *magnitude* of such induced dipole moments is typically small compared to *permanent* dipole moments, like the one we have in water. However, *qualitatively* the behaviour will be the same.
Therefore: If you have understood this exercise, you have essentially understood how a dielectric behaves in an external electric field.

The cross product between two vectors
The cross product between two vectors is a third vector with direction perpendicular to both, and with a magnitude given by the product of the magnitudes of the two, multiplied with the sine of the angle between them.
The sign of the angle between the two vectors is chosen to be positive when we go *from* the first vector *towards* the second one. This sign convention corresponds to the well-known (?) right hand rule:



Let the four fingers (not the thumb) of your right hand point along the first vector. Next, bend them until they point along the second vector. (We bend the fingers in the direction which yields an angle smaller than 180 degrees.) The thumb now points in the direction of the cross product. Thus:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$

has absolute value

$$c = |\boldsymbol{c}| = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \sin\theta = a \cdot b \cdot \sin\theta$$

Example 1: $\boldsymbol{a} = 10\,\hat{x}$ and $\boldsymbol{b} = 5\,\hat{y}$ yields

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = 50\,\hat{z}$$

Example 2: $\boldsymbol{a} = 5\,\hat{y}$ and $\boldsymbol{b} = 10\,\hat{x}$ yields

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = -50\,\hat{z}$$

From these examples, we notice that

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$$

Example 3: $\boldsymbol{a} = 2\,\hat{x} - 3\,\hat{y}$ and $\boldsymbol{b} = 5\,\hat{x} + 2\,\hat{y}$ yields

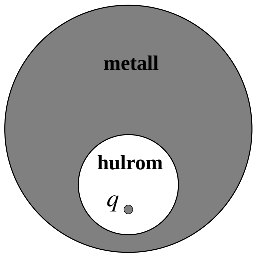
$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = 2 \cdot 2\,\hat{z} + 3 \cdot 5\,\hat{z} = 19\,\hat{z}$$

In these examples, we have used

$$\begin{aligned} \hat{x} \times \hat{x} &= 0 \\ \hat{y} \times \hat{y} &= 0 \\ \hat{x} \times \hat{y} &= \hat{z} \\ \hat{y} \times \hat{x} &= -\hat{z} \end{aligned}$$

Exercise 3

The figure below shows a cross section through the center of a metal sphere with a spherical cavity inside. (The cavity is not concentric with the metal sphere.) Inside the cavity, there is a positive point charge q (located in the cross section that goes through the center of the two spheres, but not located in the center of the cavity). The metal sphere is otherwise electrically neutral so that the net charge of the whole system is q. The point charge is fixed in the given position.



What can you then say about the distribution of (free) charge in the metal sphere, given that we have electrostatic (equilibrium) conditions? [Hint 1: What is the electric field inside the metal? Hint 2: Use Gauss' law to support your arguments.]
Sketch field lines for the electric field $\boldsymbol{E}$.
Are you able to figure out what $\boldsymbol{E}$ looks like outside the sphere? [The solution is simple. However, the explanation behind the answer is perhaps not so simple?]