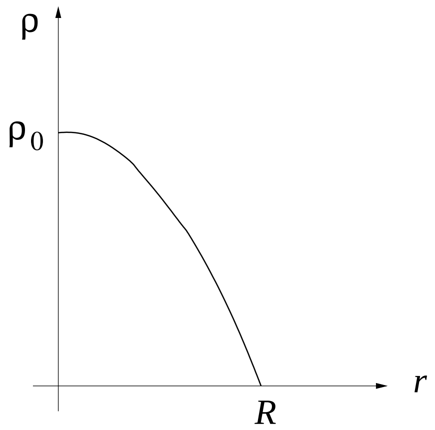
Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
Vår 2005

Solution to øving 7

Guidance February 24 and 25

Exercise 1

a) Sketch of the charge density $\rho(r)$:



A small volume element dV in position $\boldsymbol{r}$ contains a charge $dq = \rho(\boldsymbol{r})\, dV$. Here, we have a spherically symmetric charge distribution, so we may use a thin spherical shell with radius r and thickness dr as volume element: $dV = 4\pi r^2\, dr$. The total charge on the sphere is obtained by summing up the charge in such spherical shells, i.e., we must integrate from $r = 0$ to $r = R$:

$$
\begin{aligned}
Q &= \int_{r<R} \rho\, dV \\
&= 4\pi\rho_0 \int_0^R \left(1 - \frac{r^2}{R^2}\right) r^2\, dr \\
&= 4\pi\rho_0 |_0^R \left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right) \\
&= 4\pi\rho_0 R^3 \left(\frac{1}{3} - \frac{1}{5}\right) \\
&= \frac{8\pi\rho_0 R^3}{15}
\end{aligned}
$$

b) With spherically symmetric charge distribution, we choose a sphere with radius r as our Gaussian surface. The integral in Gauss' law then becomes

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = E(r) \cdot 4\pi r^2$$

If $r > R$, the whole charge $Q = 8\pi\rho_0 R^3/15$ is inside the Gaussian surface. Then, the electric field becomes

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{2\rho_0 R^3}{15\varepsilon_0 r^2}$$

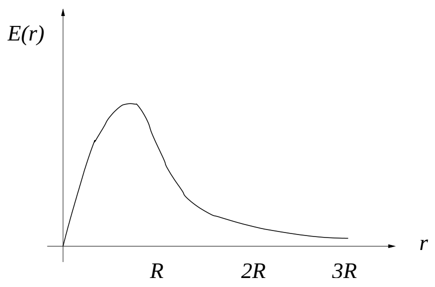
If $r < R$, the charge inside the Gaussian surface is

$$\begin{aligned} q_{\text{in}}(r) &= \int_{r'<r} \rho(r')\, dV \\ &= 4\pi\rho_0 \int_0^r \left(1 - \frac{r'^2}{R^2}\right) r'^2\, dr' \\ &= 4\pi\rho_0 |_0^r \left(\frac{r'^3}{3} - \frac{r'^5}{5R^2}\right) \\ &= 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right) \end{aligned}$$

so the electric field becomes

$$E(r) = \frac{q_{\text{in}}(r)}{4\pi\varepsilon_0 r^2} = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^3}{5R^2}\right)$$

Sketch of $E(r)$:



Exercise 2

Here, the two spheres and the metal wire is a connected electric conductor. This means that we have the same value of the electric potential on the two spheres. Since the two spheres are far away from each other, we have a spherically symmetric charge distribution on the surface of each sphere. If sphere 1 has charge Q_1, the potential on this sphere is $V_1 = Q_1/4\pi\varepsilon_0 R_1$. Let us show this: We know that the electric field outside the sphere is the same as if the whole charge Q_1 was located at the centre of the sphere, i.e., $E_1(r) = Q_1/4\pi\varepsilon_0 r^2$, directed radially outwards (if $Q_1 > 0$). With the choice $V_1 = 0$ at infinity, we find the potential on the sphere by integrating the electric field:

$$V_1(R_1) = -\int_\infty^{R_1} E_1(r)\, dr = -\int_\infty^{R_1} \frac{Q_1}{4\pi\varepsilon_0 r^2}\, dr = \frac{Q_1}{4\pi\varepsilon_0 R_1}$$

And with charge Q_2 on the other sphere, the potential there is

$$V_2(R_2) = \frac{Q_2}{4\pi\varepsilon_0 R_2}$$

Equality between these two potentials results in

$$\frac{Q_1}{Q_2} = \frac{R_1}{R_2}$$

At the same time, we must of course have $Q_1 + Q_2 = Q$, so that

$$Q_1 = \frac{R_1}{R_1 + R_2} Q \quad , \quad Q_2 = \frac{R_2}{R_1 + R_2} Q$$

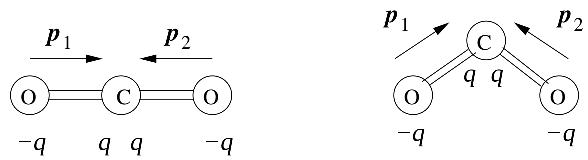
The electric field strength on the surface of sphere j ($j = 1, 2$) is

$$E_j(R_j) = \frac{\sigma_j}{\varepsilon_0} = \frac{Q_j}{4\pi\varepsilon_0 R_j^2} = \frac{Q}{4\pi\varepsilon_0 R_j(R_1 + R_2)}$$

In other words: More charge on the sphere with the bigger radius. And biggest electric field strength on the sphere with the smalles radius.

Exercise 3

We must expect that CO_2 is linear when the dipole moment is zero. Even if we don't know whether the C atom or the O atoms attract the binding electrons more strongly, we must at least expect some difference so that there is a small negative charge near the O atoms, and hence a small positive charge near the C atom, or the other way around. In real life, the former is the case, so if we "assign" a charge $-q$ to each O, we must assign a charge $+2q$ to C (since the molecule is overall electrically neutral). The total dipole moment of the molecule can then be thought of as the (vector-) sum of two dipole moments, associated with two pairs of charges q and $-q$:



We see that with a linear structure, $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$, and with a bent structure, we have $\boldsymbol{p} \neq 0$.

We may argue similarly in order to determine the structure of NH_3 and BF_3. First: From symmetry reasons, all the bond lengths N-H must be equal, and correspondingly for B-F. In addition, all bond angles H-N-H must be equal, and correspondingly for F-B-F. Then only two possibilities exist: Either the molecule is planar or it is pyramidal, with the 3 identical atoms on the "floor" and the last one on top. Also in these two molecules we must expect different *electronegativity*, i.e., "ability to attract extra electrons". In BF_3, the F becomes slightly negative ($-q$) and B slightly positive ($+3q$), in NH_3, H becomes slightly positive ($+q$) while N becomes slightly negative ($-3q$). In each molecule, we end up with a dipole moment equal to the vector sum of three dipole moments associated with three pairs of charges q and $-q$. If the molecule is planar, the sum of these three is zero, if the molecule is pyramidal, the sum is not zero. The conclusion is: BF_3 is a planar molecule and NH_3 has a pyramidal structure. And that's exactly what is found experimentally: In BF_3, the angle F-B-F is 120 degrees, in NH_3, the angle H-N-H is about 107 degrees.

Exercise 4

a) In the lectures, we found that the capacitance of a parallel plate capacitor is

$$C_0 = \varepsilon_0 \frac{A}{d}$$

where A is the plate area and d is the distance between the plates. With numbers inserted, we find

$$C_0 = 8.85 \cdot 10^{-12} \cdot \frac{0.5^2}{0.01} = 2.21 \cdot 10^{-10}$$

In the SI system, capacitance has the unit F (farad). With the prefix p for pico, which means 10^{-12}, we may write $C_0 = 221$ pF.

b) We know that we have (approximately) uniform electric field between two (approximately infinitely) large parallel plates with uniformly distributed charge of opposite sign. Hence (with V = potential difference between the plates):

$$E = \frac{V}{d} = \frac{96}{0.01} = 9600 \text{ V/m}$$

Since the capacitance, pr definition, equals the charge Q divided by the potential difference V, we have

$$Q = C_0 V = 2.21 \cdot 10^{-10} \cdot 96 = 2.12 \cdot 10^{-8} \text{ C}$$

Or, with the prefix n for nano, i.e., 10^{-9}: $Q = 21.2$ nC.

c) In the lectures, we have shown that if the volume between the plates is filled with a dielectric with relative permittivity (dielectric constant) ε_r, the capacitance is increased by a factor ε_r compared to an air-filled capacitor. Here, the plate charge Q does not change, so if the capacitance increases by a factor ε_r, the potential difference must decrease by the same factor. Hence:

$$\varepsilon_r = \frac{96}{1.2} = 80$$

The electric susceptibility χ_e equals $\varepsilon_r - 1$, so for water, $\chi_e = 79$. The permittivity of the medium (here: water) is $\varepsilon = \varepsilon_r \varepsilon_0$ so for water we have $\varepsilon = 80 \cdot 8.85 \cdot 10^{-12} = 7.08 \cdot 10^{-10}$. From Coulomb's law, we know that a possible unit for electric permittivity is C^2/Nm^2. Since capacitance has unit F, an alternative SI unit for permittivity is F/m. Both relative permittivity and electric susceptibility are dimensionless quantities.

d) The electric susceptibility χ_e was defined by the assumed linear relation between the polarization $\boldsymbol{P}$ and the electric field $\boldsymbol{E}$:

$$\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$$

Hence:

$$P = \varepsilon_0(\varepsilon_r - 1)E = \varepsilon_0(\varepsilon_r - 1)\frac{1}{\varepsilon_r}E_0 = 8.85 \cdot 10^{-12} \cdot \frac{79}{80} \cdot 9600 \simeq 8.39 \cdot 10^{-8} \text{ C/m}^2$$

Maximum theoretical polarization in water is:

$$P_{\max} = \frac{N_A \cdot p_{\text{vann}}}{v_m} = \frac{6.02 \cdot 10^{23} \cdot 6.2 \cdot 10^{-30}}{18 \cdot 10^{-6}} \simeq 0.21 \text{ C/m}^2$$

Here, $N_A = 6.02 \cdot 10^{23}$ is the socalled Avogadro number, i.e., the number of molecules in one mole, and $v_m = 18 \cdot 10^{-6}$ m^3 is the molar volume of water.
We see that the polarization P is only a small fraction of the maximum theoretical polarization P_{max}, more precisely

$$P/P_{\text{max}} \simeq 4 \cdot 10^{-7}$$

In other words, the water molecules do not exactly line up perfectly. It is more a small *tendency* to alignment of electric dipoles along the external electric field.

Exercise 5

The two narrow rings (see figure in the exercise text) in mutual distance $\boldsymbol{d} = 2z\,\hat{z} = 2R\cos\theta\,\hat{z}$ have charge $\pm dq = \pm\sigma dA = \pm\sigma \cdot (2\pi\rho) \cdot (Rd\theta) = \pm\sigma \cdot (2\pi R\sin\theta) \cdot (Rd\theta)$. The dipole moment of the sphere is therefore:

$$\begin{aligned}
\boldsymbol{p} &= \int d\boldsymbol{p} \\
&= \int \boldsymbol{d}\,dq \\
&= \int \boldsymbol{d}\,\sigma\,dA \\
&= \int_0^{\pi/2} 2z\hat{z}\sigma \cdot 2\pi R^2 \sin\theta d\theta \\
&= \int_0^{\pi/2} 2R\cos\theta\hat{z}\sigma \cdot 2\pi R^2 \sin\theta d\theta \\
&= 4\pi R^3\sigma\hat{z} \int_0^{\pi/2} \cos\theta\sin\theta d\theta \\
&= 4\pi R^3\sigma\hat{z}\,|_0^{\pi/2} \frac{1}{2}\sin^2\theta \\
&= 2\pi R^3\sigma\hat{z}
\end{aligned}$$