Institutt for fysikk, NTNU
TFY4155/FY1003: Elektrisitet og magnetisme
Spring 2005

Summary, week 18 (May 3 and 4)

Self inductance

[FGT 32.1; YF 30.2; TM 28.6; AF 27.8; LHL 25.1; DJG 7.2.3]

A current I in a current loop results in a magnetic field $\boldsymbol{B}$ in the region around it. This field, we may, at least in principle, evaluate via Biot-Savarts law:

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{l} \times \hat{r}}{r^2}$$

(see e.g. week14.pdf). The magnetic field from the loop results in a magnetic flux through the loop itself:

$$\phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{A} = \int_S \left\{ \frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{l} \times \hat{r}}{r^2} \right\} \cdot d\boldsymbol{A}$$

No matter what this integral may be, we may write

$$\phi = LI$$

assuming I is constant in the loop, which it must be, to avoid the piling up of charge somewhere in the loop...!
The factor L is the *self inductance* (or simply *inductance*) of the loop and expresses how much magnetic flux we get through the loop when there runs a current in the loop:

$$L = \frac{\phi}{I}$$

Unit for inductance: $[L] = [\phi_m/I] = [B \cdot A/I] = \mathrm{T{\cdot}m^2/A} \equiv \mathrm{H}$ (henry)

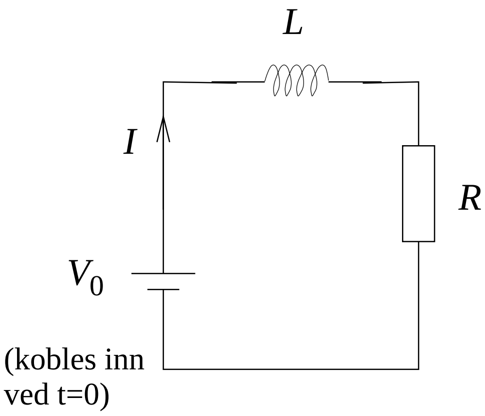
(Self-) Induction:
A time dependent current $I(t)$ in a loop results in a time dependent magnetic flux $\phi(t)$ through the loop, and hence an induced emf in the loop:

$$\mathcal{E} = -\frac{d\phi}{dt} = -L\frac{dI}{dt}$$

RL circuit

[FGT 32.4; YF 30.4; TM 28.8; AF Ex 27.5; LHL 25.2; DJG Ex 7.12]

We consider an *inductance* L (e.g. a solenoid) and a *resitance* R coupled in series in a circuit. A voltage source, e.g. a battery, V_0 is connected at time $t = 0$.



Total emf in the circuit is then

$$V_0 - L\frac{dI}{dt}$$

where the last term is the induced "back voltage" over the inductance when we try to *change* the current through it.
According to the Kirchhoff voltage rule, this total emf must correspond to the voltage drop across the resistor R, i.e.,

$$V_0 - L\frac{dI}{dt} = RI$$

or

$$L\frac{dI}{dt} + RI = V_0$$

This is precisely the same type of 1. order differential equation for the current I as we had for the charge Q when we studied charging of a capacitor in an RC circuit (see week13.pdf, page 4). The solution is

$$I(t) = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)$$

where we have used the initial condition $I(0) = 0$. (Before connecting the battery, we obviously have $I = 0$. At time $t = 0$, the current cannot suddenly "jump" to a finite value different from zero. Why not? Well, that would imply that $dI/dt \to \infty$ in $t = 0$, which would again mean an infinite back voltage across the inductance. And that is not physically possible! So, I must be continuous in $t = 0$, and we may put $I(0) = 0$.)

Time constant for the change in the current:

$$\tau = \frac{L}{R}$$

The value of τ provides an "order of magnitude" for how long time it takes to increase the current in an RL circuit from zero to its maximal value V_0/R:

$$I(t \to \infty) = \frac{V_0}{R}$$

Energy in magnetic field
[FGT 32.2, 32.3; YF 30.3; TM 28.7; AF 26.8, 27.11; LHL 25.3; DJG 7.2.4]

Let us find out how much energy we must supply to a solenoid with inductance L when we increase the current from $i = 0$ to a final value $i = I$.
Supplied energy when increasing the current from i to $i + di$:

$$dU_B = P\ dt = iv\ dt = iL\frac{di}{dt}dt = Li\ di$$

Here, $P = iv$ is the supplied power, i.e., the supplied energy pr unit time, and $v = Ldi/dt$ is the voltage across the solenoid when we change the current from i to $i + di$.
So, the total energy supplied in order to increase the current from 0 to I is

$$U_B = \int dU_B = L\int_0^I i\ di = \frac{1}{2}LI^2$$

This energy can now be associated with the magnetic field B inside the solenoid. Assume that the solenoid is very long, with N turns along the full length l. The cross section has area A. Then, the magnetic field inside the solenoid is

$$B = \mu_0 nI = \mu_0\frac{N}{l}I$$

Outside, it is zero. The total magnetic flux through all the N turns becomes

$$\phi_m = NAB = NA\mu_0\frac{N}{l}I$$

which we may write as

$$\phi_m = LI$$

where L is the inductance of the solenoid. With this, we may rewrite the expression for the energy U_B:

$$U_B = \frac{1}{2}\frac{NAB}{I}I^2 = \frac{1}{2}NAB\cdot\frac{Bl}{\mu_0 N} = \frac{1}{2\mu_0}B^2\cdot Al$$

Here, Al is the volume inside the solenoid, so we see that we have an *energy density* (i.e. energy pr unit volume) associated with the magnetic field B:

$$u_B = \frac{1}{2\mu_0}B^2$$

Earlier, we found that we have an electrical energy density u_E associated with an electric field E:

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

Hence, the *total energy density in an electromagnetic field* is:

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

Comment: This expression is always correct, in the sense that u represents the energy stored in the field E and B. In the literature, you may come across the formula

$$u = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E} + \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{H} = \frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}B^2$$

for the total energy density if you have polarizable and/or magnetizable media present. (Here, we used $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, with ε = the permittivity of the medium and μ = the permeability of the medium.)
These two expressions for u are not identical, and cannot therefore represent the same energy density. The latter expression for u includes a contribution which is not directly stored in the fields, namely the "elastic" energy associated with the polarization and magnetization, i.e., the alignment of electric and magnetic dipoles.
To the extent that this is relevant for the exam, we will exclusively worry about the *field energy*, given by

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

LC **circuit**
[FGT 32.5; YF 30.5; TM 29.5; AF 27.9; LHL 27.1; DJG Problem 7.25]

Kirchhoff's voltage rule gives

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0$$

where the first term is the induced emf in the inductor and the second term is the voltage drop in the capacitor; Q and I are the charge on the capacitor and the current in the circuit, respectively. Insertion of $I = dQ/dt$ yields

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

which has the solution

$$Q(t) = \tilde{Q}\cos(\omega_0 t + \alpha)$$

The angular frequency ω_0 is found by insertion:

$$-\omega_0^2 Q + \frac{1}{LC}Q = 0$$

i.e.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The two integration constants $\tilde{Q}$ and α are determined with two initial conditions. For example: $Q(0) = Q_0$ and $I(0) = 0$. That results in $\alpha = 0$ and $\tilde{Q} = Q_0$ so that

$$\begin{aligned} Q(t) &= Q_0 \cos\omega_0 t \\ I(t) &= -\omega_0 Q_0 \sin\omega_0 t = \omega_0 Q_0 \cos(\omega_0 t + \pi/2) \end{aligned}$$

In other words: The charge $Q(t)$ on the capacitor and the current $I(t)$ in the circuit both have a *harmonic* time dependence, with a mutual *phase difference* of $\pi/2$, i.e., 90 degrees.
We have energy conservation:
Electric energy stored in the electric field between the capacitor plates:

$$U_E = \frac{1}{2C}Q^2 = \frac{1}{2C}Q_0^2\cos^2\omega_0 t$$

Magnetic energy stored in the magnetic field in the inductor (the solenoid):

$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}L\omega_0^2 Q_0^2 \sin^2 \omega_0 t = \frac{1}{2}L \cdot \frac{1}{LC} \cdot Q_0^2 \sin^2 \omega_0 t = \frac{1}{2C}Q_0^2 \sin^2 \omega_0 t$$

Total energy in the circuit:

$$U = U_E + U_B = \frac{1}{2C}Q_0^2$$

which is constant.

Ampere-Maxwell's law

Not required knowledge on the exam, but see separate "note" on the homepage.