Institutt for fysikk, NTNU
TFY4155/FY1003: Elektrisitet og magnetisme
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Summary, week 7 (February 16)

When is Gauss' law particularly useful?

[FGT 23.3; YF 22.4; TM 22.3; AF 25.4; LHL 19.7; DJG 2.2.3]

Gauss' law can be used to determine the electric field from a charge distribution where we have some kind of *symmetry*. Examples: Spherical, planar or cylindrical symmetry.

Cylindrical symmetry: See øving 5, infinitely long rod with uniform charge λ pr unit length.

Spherical symmetry: In the lectures, we looked at a uniformly charged spherical surface and found that the electric field inside the sphere was zero, whereas the field outside the sphere was as if the whole charge Q was located in the centre of the sphere, i.e.,

$$E(r) = Q/4\pi\varepsilon_0 r^2$$

When we have a spherically symmetric charge distribution, we must realize that the electric field has to be radially directed, and furthermore that the electric field strength $E = |\boldsymbol{E}|$ only depends on the distance r from the centre of the charge distribution (and not the angles θ and ϕ, i.e., where we are on a given spherical surface). Then we realize that it is smart to choose precisely a spherical surface with radius r as our Gaussian surface in order to find $E(r)$, because the "surface element vector" $d\boldsymbol{A}$ is now parallel to $\boldsymbol{E}$ all over the Gaussian surface. The integral in Gauss' law is now easy to perform:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = E(r) \cdot 4\pi r^2$$

With a spherically symmetric charge distribution, i.e., the charge pr unit volume ρ is only a function of r' (and not θ' and ϕ'), it is also easy to determine how much charge we have inside the Gaussian surface:

$$\begin{aligned} q_{\text{in}}(r) &= \int_{r'<r} \rho(r')dV' \\ &= \int_{r'=0}^{r} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \rho(r')(r')^2 \, dr' \sin\theta' \, d\theta' \, d\phi' \\ &= 4\pi \int_{r'=0}^{r} \rho(r') \, (r')^2 \, dr' \end{aligned}$$

Provided the charge density $\rho(r')$ isn't a "too awful" mathematical function, we are often able to solve the resulting integral over r'. For example, we notice that if ρ is constant, we obtain $q_{\text{in}}(r) \sim r^3$, and if $\rho(r') \sim r'$ (i.e., linearly increasing), we have $q_{\text{in}}(r) \sim r^4$. In the former case, the result is $E(r) \sim r$, in the latter case, $E(r) \sim r^2$.
Note that if r is sufficiently large that all of the charge Q lies inside the Gaussian surface (i.e.: $\rho(r') = 0$ for $r' > r$), we will always find $E(r) = Q/4\pi\varepsilon_0 r^2$, i.e., as if the whole charge was located at the centre. This only results when we have spherical symmetry.

Planar symmetry: Another important example is the uniformly charged, infinitely large plane. We showed that if the plane has charge σ pr unit area, the electric field is

$$E = \frac{\sigma}{2\varepsilon_0}$$

i.e., independent of the distance to the charged plane. In this example, we realize, first, that $\boldsymbol{E}$ must everywhere be perpendicular to the charged plane, and, second, that the field strength $E = |\boldsymbol{E}|$ cannot possibly depend on anything but the distance to the plane. Taking this as our starting point, we realize that a smart Gaussian surface must be a pill box or cylinder that is cut in the middle by the charged plane. Then, we have the surface element vector $d\boldsymbol{A}$ parallel to $\boldsymbol{E}$ on the two parts of the Gaussian surface that are parallel with the charged plane, whereas $d\boldsymbol{A}$ is perpendicular to $\boldsymbol{E}$ on the rest of the Gaussian surface. Hence, the integral in Gauss' law reduces to $2EA$, where A is the area of the "end surface" of the Gaussian surface. The charge inside such a Gaussian surface is σA, because an area A of the charged plane now lies inside the Gaussian surface. So, we found that E was simply constant, and not even dependent upon the distance to the charged plane!

Having the latter result, it was easy to find the electric field due to *two* parallel planes with opposite charge, $\pm\sigma$ (both planes are infinitely large). Using the superposition principle, we found

$$E = \frac{\sigma}{\varepsilon_0}$$

in the region between the two planes, and

$$E = 0$$

in the region outside the planes. Between the planes, the field is directed from the positive towards the negative plane.
The latter example is very relevant: It represents a socalled *parallel plate capacitor*, where the area of the plates is large compared to the distance between them. We will come back to this example many times.

Materials and their electric properties

Main categories of materials with respect to electric properties:

Conductors: Metals. The outermost electron(s) on an atom are *free* to move through the material. Examples: Cu, Al, Ag etc.

Insulators: All electrons are *bound* to the atoms. Examples: glass, plastics, wood etc.

Semiconductors: These are insulators at $T = 0$ (i.e., zero temperature), but some electrons can be made free if $T > 0$. Important materials in electronic equipment (e.g. diodes, transistors). Examples: Si, Ge, GaAs.

Furthermore: Superconductors, plasmas, electrolytes etc.

Comment: Metals are *good* conductors. Some other materials (e.g. graphite) are *poorer* conductors. Later, when we come to electric current and electric circuits, we will quantify this in terms of a material's *conductivity.* Insulators like plastics, glass and wood have very poor conductivity. It is not precisely zero, but many orders of magnitude smaller than the conductivity of both good and poor conductors. More about this later.

In this course, we will only deal with conductors and insulators. An insulator is also called a *dielectric.* We will first talk about conductors.

Conductors

[FGT 23.4; YF 21.2, 22.5; TM 21.2, 22.5; AF 25.5; LHL 19.2, 19.8; DJG 2.5]

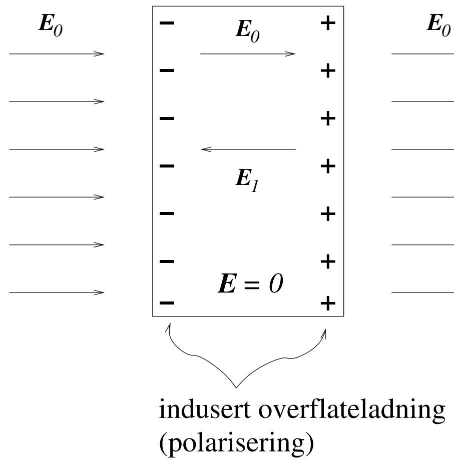
In the lectures, we proved the following 7 important results concerning electric field, potential and charge distribution on a conductor in *electrostatic equilibrium*:

1. Inside the conductor, $\boldsymbol{E} = 0$.
2. There is zero net charge inside the conductor.
3. All the net charge resides on the surface of the conductor.
4. On the surface of the conductor, the electric field is perpendicular to the surface.
5. The complete conductor, both inside and the surface, has one and the same value of the electric potential V, i.e., the conductor is an *equipotential.*
6. On the surface of the conductor, the electric field is $E = \sigma/\varepsilon_0$, where σ is the charge pr unit area on the surface of the conductor.
7. A conductor with an empty cavity has $E = 0$ inside the cavity, and all the net charge on the outer surface of the conductor. (Of course, this is not the case if we do have a net charge within the cavity, e.g., a point charge.)

Comment: We have used Gauss' law in order to prove that all the net charge must reside on the surface of a conductor. However, we could also have argued in terms of energy: The net charge must distribute itself so as to minimize the potential energy of the system. It may not be completely obvious that the lowest energy is obtained with all the net charge on the surface. Perhaps we could "earn" a bit by putting some of the charge around inside the volume of the conductor? Well, that turns out *not* to be the case! Any conductor, regardless the size and the shape, will always have the minimum potential energy when all the net charge distributes itself on the surface.

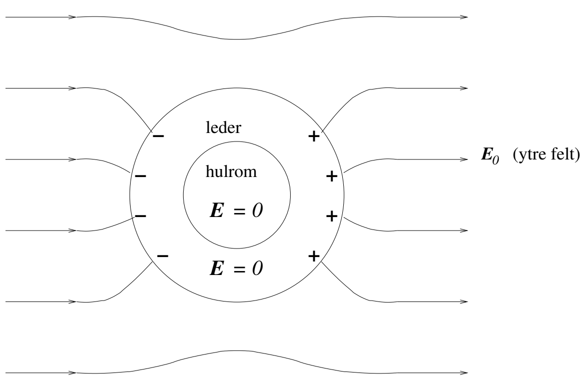
Conductor in external field

An external field $\boldsymbol{E}_0$ will affect free ("mobile") charges in the conductor so that electric forces act on them. Some of these charges will then move to the surface of the conductor so that the *induced field* $\boldsymbol{E}_1$ exactly cancels the external field inside the conductor: $\boldsymbol{E}_1 = -\boldsymbol{E}_0$. Hence, the total field inside the conductor is $\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1 = 0$, as we have proven that it must be.



Conductor with cavity in external field

Electrostatic field inside cavity of a conductor in external field $\boldsymbol{E}_0$ is zero:



I.e.: Inside the cavity, we are electrostatically *screened* from the external field.
In order to have perfect screening, we must have a "genuine" cavity, i.e., a specific volume, as inside a soccer ball. In practice, we obtain very good screening even with openings from the outside and into the cavity. Cf. the Faraday cage used on the lab. Other examples are cars and airplanes.

Note that a conductor screens an external field so that $E = 0$ inside an empty cavity. However, the opposite is not the case: The conductor does not screen the field from a charge put inside the cavity. If we have a charge inside the cavity, we will have $E \neq 0$, both inside the cavity and outside the conductor. (But $E = 0$ inside the conductor, of course. See øving 6.)