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Mechanical energy and momentum of wave pulses in a dispersionless, lossless elastic medium, according to the linear theory

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We re-examine the linear theory of wave propagation through an elastic string under uniform tension or a slender elastic rod from a perspective that focuses on the flow of mechanical energy and mechanical momentum. Continuity equations are established for the flow of energy and momentum, leading to two boundary conditions for the net wave displacement. The important special case of a small amplitude pulse of arbitrary shape traveling through a uniform slender medium joined to another medium with a different linear mass density is examined in detail. The new boundary conditions lead to the correct relative amplitudes for the reflected and transmitted pulses. We obtain the instantaneous mechanical energy and momentum of the incident, reflected, and transmitted pulses and show that the net mechanical energy and momentum are separate constants of motion. The cases of an incoming pulse described by a Lorentzian and a Gaussian distribution are suggested as problems to be solved by the interested reader.
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I. INTRODUCTION

If there is no dispersion or dissipation, a wave pulse traveling on a taut elastic string or a slender elastic rod carries with it a net mechanical energy and a net mechanical momentum that are constant over time. In lowest-order or linear approximation for the wave equation, the standard expressions for the mechanical energy and the mechanical momentum densities are[1–3]

$$U_E(x,t)=\frac{1}{2}\rho(x)\left[\left(\frac{\partial u(x,t)}{\partial t}\right)^2+\left(\nu(x)\frac{\partial u(x,t)}{\partial x}\right)^2\right], \tag{1}$$

and

$$U_P(x,t)=\rho(x)\frac{\partial u(x,t)}{\partial t}, \tag{2}$$

where x is the position, t is the time, and $u(x,t)$ is the local displacement of the medium with respect to its equilibrium position. Also, $\rho(x)$ is the equilibrium linear mass density of the medium and $\nu(x)$ is the velocity of the wave at x. For waves on a string, u is taken to be purely transverse to the x axis, while for waves in a rod, u is assumed to be purely longitudinal. The quantity,

$$F=\rho(x)\nu(x)^2, \tag{3}$$

is assumed to be constant over time and independent of x. For transverse waves on a string, F is the equilibrium tension; for longitudinal waves in an elastic rod, F represents the Young's modulus of the material. We will not consider situations where F varies with the position in the medium such as a rope that hangs vertically in a gravitational field.

Nonlinearities are intrinsic to any vibrating elastic medium. When the nonlinearities are large, they can give rise to important coupling between the tranverse and the longitudinal modes of motion of the medium in which they travel.[4] As a result, energy and momentum can be exchanged between corresponding modes of motion of the vibrating medium. However, we will assume that the nonlinearities are negligible for the pulse amplitudes that are of interest so their effects will not be considered. Readers interested in nonlinear wave theory should consult Ref. 5, for example.

Equation (1) for the mechanical energy density is well known and used often.[1–3,5] Although Eq. (2) for the mechanical momentum density also is well known,[1–3] it is not often used. For example, Corben and Stehle[1] make extensive use of Eq. (1) in their treatment of the stretched string, but they merely observe that the mechanical momentum density ''will average to zero'' in time and that because the ''——term vanishes either in the mean or completely, we ignore it henceforth——.'' Morse and Feshbach[2] and Goldstein[3] also introduced Eq. (2) and referred to the mechanical momentum density, but did not discuss it any further in the context of the linear wave equation. Gurevich and Thellung[6] defined the density of ''ordinary momentum'' as in Eq. (2), but then argued, consistent within Ref. 1, that the net mechanical momentum is of little interest because it vanishes, whereas the space integral of the energy density does not. As a result, the mechanical energy of the system is an ''interesting integral of the motion'' whereas that for the mechanical momentum is not.[1]

Gilbert and Mollow[7] are a welcome exception and considered the transport of mechanical momentum by a purely longitudinal wave in a slender elastic rod. Those authors started from Eq. (2) and took the total momentum of the rod to be the sum of the momenta of the individual particles. They then showed that when an external particle is coupled to the rod, ''——The sum of the momentum of the particle plus the ordinary momentum of the rod is ... conserved.——'' The expression ''ordinary momentum'' refers to the space integral of Eq. (2), that is, the net mechanical momentum of the wave.

The goal of this article is to examine, in more detail than in previous studies, the consequences of Eq. (2) for the linear theory of mechanical waves in dispersionless and lossless media. It will be shown that Eqs. (1) and (2) lead to a consistent theory for the propagation of mechanical waves in a linear medium. We also show that this theory is compatible with conservation of the total mechanical energy and total mechanical momentum. In our approach, the mechanical momentum density, Eq. (2), is not an insignificant quantity. On the contrary, it is as important as the mechanical energy density, Eq. (1).

The article is organized as follows. The two equations of continuity are obtained in Sec. II. The important case of two uniform media with different linear mass densities joined to one another is examined in Sec. III. We show that a consequence of the continuity equations for the energy and momentum flow is the correct boundary conditions for the wave amplitudes. These boundary conditions are usually derived in an ad hoc manner by requiring that the net displacement and its gradient be continuous across the junction.[8,9] In the present treatment these extra assumptions are not required, instead they emerge naturally from the equations of continuity for the energy and momentum. Other consequences of the present approach are results for the mechanical energy and momentum of the incident, reflected, and transmitted pulses as functions of the time. It will then be shown that the net mechanical energy and net momentum of this three-pulse wave system are separately conserved. The specific cases of Lorentzian- and Gaussian-shaped pulses are suggested as exercises for the interested reader.

II. CONTINUITY EQUATIONS FOR THE FLOW OF MECHANICAL ENERGY AND MOMENTUM

The linear form of the classical wave equation in a lossless and dispersionless medium is

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where v^2 is related by Eq. (3) to the linear mass density of the medium and the uniform equilibrium tension or the uniform Young's modulus. As a result, $v=v(x)$ may be a function of x, but not of t.

We start by deriving the continuity equation for the energy. We take the partial derivative of Eq. (1) with respect to time and use Eq. (4) to find

$$\begin{aligned}\frac{\partial U_E}{\partial t}&=\rho\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}+\rho v^2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\\&=\rho\left(\frac{\partial u}{\partial t}\right)\left(v^2\frac{\partial^2 u}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(\rho v^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)\\&\quad-\frac{\partial}{\partial x}\left(\rho v^2\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}\\&=F\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)-F\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\\&=\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right).\end{aligned} \tag{5}$$

The uniformity of $\rho v^2=F$ was used. We then have

$$\frac{\partial U_E}{\partial t}+\frac{\partial j_E}{\partial x}=0, \tag{6}$$

where

$$j_E\equiv -F\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} \tag{7}$$

represents the current of mechanical energy at time t at position x in the medium. Equation (6) is the usual form for a continuity equation and it governs the flow of mechanical energy in the medium.

Unless there is an external sink or source of mechanical energy at x, j_E must be continuous at x. If we assume that there is no such external coupling at any point within the medium, we have

$$\lim_{\epsilon\to 0+} j_E(x-\epsilon,t)=\lim_{\epsilon\to 0+} j_E(x+\epsilon,t), \tag{8}$$

for all x and t.

Equation (8) is valid within the linear theory of one-dimensional mechanical waves and thus applies at a boundary where the linear mass density may exhibit a sudden change. At such a boundary, we have

$$\left.\frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t}\right|_{\mathrm{LB}}=\left.\frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t}\right|_{\mathrm{RB}}, \tag{9}$$

because F is assumed to be uniform. The subscripts LB and RB signify the left and right boundaries, respectively. Equation (9) represents a boundary condition on the wave amplitudes.

We now turn to the flow of mechanical momentum. We consider the partial time derivative of Eq. (2) and use Eqs. (4) and (3) to obtain

$$\frac{\partial U_P}{\partial t}=\rho\frac{\partial^2 u}{\partial t^2}=\rho v^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(F\frac{\partial u}{\partial x}\right). \tag{10}$$

This result also can be expressed in the form of a continuity equation:

$$\frac{\partial U_P}{\partial t}+\frac{\partial j_P}{\partial x}=0. \tag{11}$$

By definition,

$$j_P\equiv -F\frac{\partial u}{\partial x} \tag{12}$$

is the current associated with the flow of mechanical momentum through the medium. Equation (11) controls the flow of mechanical momentum. Consequently, in the absence of an external sink or source of momentum at x, j_P will be continuous at x, that is,

$$\lim_{\epsilon\to 0+} j_P(x-\epsilon,t)=\lim_{\epsilon\to 0+} j_P(x+\epsilon,t). \tag{13}$$

Equation (13) is valid for the linear theory of one-dimensional mechanical waves for materials that are subjected to uniform tension or are characterized by a uniform Young's modulus and is valid at a boundary between two uniform media with different linear mass densities. Hence, as in Eq. (9), we have

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{\mathrm{LB}}=\left.\frac{\partial u(x,t)}{\partial x}\right|_{\mathrm{RB}}. \tag{14}$$

Equation (14) represents a second boundary condition on the wave displacements.

The substitution of Eq. (14) into Eq. (9) gives

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{\mathrm{LB}}=\left.\frac{\partial u(x,t)}{\partial t}\right|_{\mathrm{RB}}. \tag{15}$$

Equations (14) and (15) form the complete set of independent boundary conditions for the wave amplitudes in a specific situation. Note that Eq. (14) is identical to the boundary condition that is obtained by arguing that, for transverse waves, the slopes of the string must match across the boundary to avoid having the mass element at x move laterally; see Ref. 8, for example. Unfortunately, this kind of reasoning is not as convincing for longitudinal waves in a slender rod: there is a necessity for lateral motion. Also note that Eq. (15) relates the local time rate of change of the net displacements, whereas the usual requirement is that the net amplitudes be continuous across the junction. It is easily shown by integrating over time, however, that these requirements are equivalent. The time integral of Eq. (15) indicates that $u(x,t)|_{\mathrm{LB}} - u(x,t)|_{\mathrm{RB}}$ and $u(x,t_0)|_{\mathrm{LB}} - u(x,t_0)|_{\mathrm{RB}}$ must be equal at all times. Consequently, if it is assumed that the net displacements across the boundary are equal at some initial time t_0, then these displacements must be equal for any subsequent time t.

III. ENERGY AND MOMENTUM FOR A SYSTEM WITH A JUNCTION

Consider a junction at $x=0$ between two slender, uniform media. The linear mass density for this junction is

$$\begin{aligned}\rho(x) &= \rho_1, \quad -\infty \leqslant x \leqslant 0,\\ &= \rho_2, \quad 0 < x \leqslant +\infty,\end{aligned} \tag{16}$$

where ρ_1 and ρ_2 are uniform and constant. A small amplitude wave pulse described by

$$u_I(x,t) = f(t - x/\nu_1), \tag{17}$$

is sent from $x = -\infty$ at time $t \to -\infty$ and eventually interacts with the junction. As is well known, general solution of the wave equation for this case gives one reflected and one transmitted wave pulse and can be represented as[8]

$$u_R(x,t) = A_R f(t + x/\nu_1), \tag{18}$$

$$u_T(x,t) = A_T f(t - x/\nu_2), \tag{19}$$

respectively, where ν_1 and ν_2 are the phase velocities in the corresponding media; A_R and A_T are unknown amplitudes to be determined from the boundary conditions at the junction.

Equations (17)–(19) obviously satisfy the wave equation, Eq. (4), and allow us to find the unknown amplitudes, A_R and A_T. To do so, note that

$$\begin{aligned}\left.\frac{\partial f(t \pm x/\nu_1)}{\partial x}\right|_{x=0^-} &= \pm \frac{1}{\nu_1} \left.\frac{\partial f(t \pm x/\nu_1)}{\partial t}\right|_{x=0^-}\\ &= \pm \frac{1}{\nu_1} \frac{df(t)}{dt},\end{aligned} \tag{20}$$

$$\begin{aligned}\left.\frac{\partial f(t - x/\nu_2)}{\partial x}\right|_{x=0^+} &= -\frac{1}{\nu_2} \left.\frac{\partial f(t - x/\nu_2)}{\partial t}\right|_{x=0^+}\\ &= -\frac{1}{\nu_2} \frac{df(t)}{dt}.\end{aligned} \tag{21}$$

With the help of these results, Eq. (14) gives

$$1 - A_R = \frac{\nu_1}{\nu_2} A_T. \tag{22}$$

Similarly, Eq. (15) gives

$$1 + A_R = A_T. \tag{23}$$

Equations (22) and (23) represent the same relation between the amplitudes as that given by the standard approach and are readily solved to give

$$A_R = \left(1 - \frac{\nu_1}{\nu_2}\right) \Big/ \left(1 + \frac{\nu_1}{\nu_2}\right), \quad A_T = 2 \Big/ \left(1 + \frac{\nu_1}{\nu_2}\right), \tag{24}$$

which are the usual results for the reflected and transmitted pulses, respectively.

We now calculate the mechanical momentum and energy associated with each pulse. Then we will show that the algebraic sum of the individual momenta and of the individual energies are two separate constants of motion, as expected.

The mechanical momentum of each pulse is given by integrating Eq. (2) as follows:

$$\begin{aligned}P_I(t) &= \int_{-\infty}^{0} dx \rho_1 \frac{\partial f(t - x/\nu_1)}{\partial t}\\ &= -\rho_1 \nu_1 \int_{\infty}^{t} dz \frac{df(z)}{dz} = \rho_1 \nu_1 [f(\infty) - f(t)],\end{aligned} \tag{25}$$

$$\begin{aligned}P_R(t) &= \int_{-\infty}^{0} dx \rho_1 A_R \frac{\partial f(t + x/\nu_1)}{\partial t}\\ &= \rho_1 \nu_1 A_R \int_{-\infty}^{t} dz \frac{df(z)}{dz} = \rho_1 \nu_1 A_R [f(t) - f(-\infty)],\end{aligned} \tag{26}$$

$$\begin{aligned}P_T(t) &= \int_{0}^{\infty} dx \rho_2 A_T \frac{\partial f(t - x/\nu_2)}{\partial t}\\ &= -\rho_2 \nu_2 A_T \int_{t}^{-\infty} dz \frac{df(z)}{dz}\\ &= \rho_2 \nu_2 A_T [f(t) - f(-\infty)].\end{aligned} \tag{27}$$

A change of variable of the type $z = t \pm x/\nu$, with time t assumed to be finite, was made to obtain Eqs. (25)–(27). It is clear that the momentum of each wave pulse varies explicitly with the time. We now show that the net instantaneous momentum is a constant of motion.

At time t, the net momentum, $P_N(t)$, is the algebraic sum of the instantaneous momenta of the three individual pulses, as given in Eqs. (25)–(27). We have that

$$\begin{aligned}P_N(t) &\equiv P_I(t) + P_R(t) + P_T(t)\\ &= \rho_1 \nu_1 \Big[(f(\infty) - f(t)) + A_R (f(t) - f(-\infty))\\ &\quad + \frac{\rho_2 \nu_2}{\rho_1 \nu_1} A_T (f(t) - f(-\infty)) \Big]\\ &= \rho_1 \nu_1 \Big[f(\infty) - f(-\infty) + (f(-\infty)\\ &\quad - f(t)) \left(1 - A_R - \frac{\nu_1}{\nu_2} A_T\right) \Big]\\ &= \rho_1 \nu_1 [f(\infty) - f(-\infty)]\end{aligned} \tag{28}$$

where

$$\frac{\rho_2 \nu_2}{\rho_1 \nu_1} = \frac{\nu_1}{\nu_2}, \tag{29}$$

and

$$1 - A_R = \frac{\nu_1}{\nu_2} A_T \tag{30}$$

were used; see Eqs. (3) and (22), respectively. The total mechanical momentum of the system is equal to a constant and is thus conserved.

We now turn to the energy of the individual pulses, $E_I(t)$, $E_R(t)$, and $E_T(t)$, and obtain the total instantaneous mechanical energy of the entire system, $E_N(t)$. First consider the total energy in the system in the region $x<0$:

$$\begin{aligned} E_{x<0}(t) &= \frac{1}{2}\int_{-\infty}^{0} dx \rho \left[\left(\frac{\partial u}{\partial t}\right)^2 + \nu^2 \left(\frac{\partial u}{\partial x}\right)^2 \right] \\ &= \frac{1}{2}\int_{-\infty}^{0} dx \rho_1 \left[\left(\frac{\partial u_I}{\partial t} + \frac{\partial u_R}{\partial t}\right)^2 + \nu_1^2 \left(\frac{\partial u_I}{\partial x} + \frac{\partial u_R}{\partial x}\right)^2 \right] \\ &= \frac{1}{2}\int_{-\infty}^{0} dx \rho_1 \left[\left(\frac{\partial u_I}{\partial t}\right)^2 + 2\left(\frac{\partial u_I}{\partial t}\right)\left(\frac{\partial u_R}{\partial t}\right) + \left(\frac{\partial u_R}{\partial t}\right)^2 \right. \\ &\quad \left. + \nu_1^2 \left(\left(\frac{\partial u_I}{\partial x}\right)^2 + 2\left(\frac{\partial u_I}{\partial x}\right)\left(\frac{\partial u_R}{\partial x}\right) + \left(\frac{\partial u_R}{\partial x}\right)^2 \right) \right]. \end{aligned} \tag{31}$$

Note from Eqs. (17) and (18) that

$$\frac{\partial u_I}{\partial t} = -\nu_1 \frac{\partial u_I}{\partial x}, \quad \frac{\partial u_R}{\partial t} = +\nu_1 \frac{\partial u_R}{\partial x}. \tag{32}$$

We find that

$$E_{x<0}(t) = \int_{-\infty}^{0} dx \rho_1 \left[\left(\frac{\partial u_I}{\partial t}\right)^2 + \left(\frac{\partial u_R}{\partial t}\right)^2 \right]. \tag{33}$$

The cross terms represent the effects of interference of the two pulses. The energy is unaffected by the process, in contrast with the strong effects of this interference on the net local displacement of the medium.

The energy of each pulse is thus

$$E_I(t) = \int_{-\infty}^{0} dx \rho_1 \left(\frac{\partial u_I}{\partial t}\right)^2 = \rho_1 \nu_1 [I_0 - I_1(t)], \tag{34}$$

$$E_R(t) = \int_{-\infty}^{0} dx \rho_1 \left(\frac{\partial u_R}{\partial t}\right)^2 = \rho_1 \nu_1 A_R^2 [I_0 + I_1(t)], \tag{35}$$

$$E_T(t) = \int_{0}^{\infty} dx \rho_2 \left(\frac{\partial u_T}{\partial t}\right)^2 = \rho_2 \nu_2 A_T^2 [I_0 + I_1(t)], \tag{36}$$

after introducing $z = t \pm x/\nu$ for each case, as done for the momentum integrals. By definition,

$$I_0 \equiv \int_0^{\infty} dz \left(\frac{df(z)}{dz}\right)^2 \tag{37}$$

$$I_1(t) \equiv \int_0^{t} dz \left(\frac{df(z)}{dz}\right)^2. \tag{38}$$

The energy of each pulse varies in time. We now show that the total instantaneous energy is a constant of motion.

At time t, the total mechanical energy of the pulses is

$$\begin{aligned} E_N(t) &\equiv E_I(t) + E_R(t) + E_T(t) \\ &= \rho_1 \nu_1 \left[(I_0 - I_1(t)) + A_R^2 (I_0 + I_1(t)) \right. \\ &\quad \left. + \frac{\rho_2 \nu_2}{\rho_1 \nu_1} A_T^2 (I_0 + I_1(t)) \right] \\ &= \rho_1 \nu_1 \left[I_0 \left(1 + A_R^2 + \frac{\nu_1}{\nu_2} A_T^2 \right) + I_1(t) \right. \\ &\quad \left. \times \left(-1 + A_R^2 + \frac{\nu_1}{\nu_2} A_T^2 \right) \right] = 2 \rho_1 \nu_1 I_0, \end{aligned} \tag{39}$$

where Eqs. (22), (23), and (29) were used. The sum of the individual energies is a constant: the net energy is thus conserved.

We now suggest two specific distributions for solution by the interested reader and present them as exercises.

Let the incoming pulse be described by the following shape-functions:

$$f(z) = I \frac{\gamma^2}{z^2 + \gamma^2} \quad (\text{Lorentzian}), \tag{40}$$

$$f(z) = I e^{-\sigma^2 z^2/2} \quad (\text{Gaussian}). \tag{41}$$

The parameters γ and σ characterize the width of each pulse and I determines the maximum value of the incident pulse amplitudes. These shapes are of particular interest because the time-dependent part of the momentum and of the energy integrals can be evaluated exactly for the incident, reflected, transmitted, and net pulses.

Problem 1: Use the above shapes to show that the integrals of Eqs. (37) and (38) are as follows:

$$I_0 = \frac{\pi I^2}{8 \gamma}, \tag{42}$$

$$\begin{aligned} I_1(t) &= \frac{I^2}{4} \left[\frac{t}{(t^2 + \gamma^2)} \left(1 + \frac{2\gamma^2}{3(t^2+\gamma^2)} - \frac{8\gamma^4}{3(t^2+\gamma^2)^2} \right) \right. \\ &\quad \left. + \frac{1}{\gamma} \tan^{-1}\left(\frac{t}{\gamma}\right) \right], \end{aligned} \tag{43}$$

for the Lorentzian, and

$$I_0 = I^2 \sigma^4 \int_0^{\infty} dz z^2 e^{-\sigma^2 z^2} = \frac{I^2 \sigma \sqrt{\pi}}{4}, \tag{44}$$

$$I_1(t) = I^2 \sigma \frac{\sqrt{\pi}}{4} \left[\operatorname{erf}(\sigma t) - \frac{2\sigma t}{\sqrt{\pi}} e^{-\sigma^2 t^2} \right], \tag{45}$$

for the Gaussian. In Eq. (45), erf(z) is the error function.

Problem 2: Obtain explicit expressions for the momentum and the energy contributions of each individual pulse (incident, reflected, etc.), as a function of time.

IV. CONCLUDING REMARKS

We have re-examined the linear theory of wave propagation in an elastic string under uniform tension or in a slender elastic rod characterized by a uniform Young's modulus. Our perspective differs from the standard approach, in that we first established two continuity equations, one for the mechanical energy and the other for the mechanical momentum. It was shown that these continuity equations lead directly to the boundary conditions for the wave amplitudes. The first of these conditions is that the gradients of the net wave displacement must match across a boundary. This condition is identical to that obtained by the standard approach. The second condition states that the local time rate of change of the net wave displacement must be equal across the boundary. This condition differs from the condition that is normally used, that is, the net displacements of the wave must be equal across a boundary. However, it was shown that these statements, although different, are fully compatible with one another.

The case of a small-amplitude incident pulse of arbitrary shape at a junction between two media with different linear mass density also was examined. The new boundary conditions were shown to lead to the correct relative amplitudes for the reflected and for the transmitted pulses. Explicit expressions for the instantaneous mechanical energy and for the instantaneous mechanical momentum of the incident, reflected, and transmitted pulses also were obtained, and the net mechanical energy and the net mechanical momentum of this three-pulse system were shown to be separate constants of motion.

We hope that the present approach will encourage authors and teachers to discuss a subject that has been neglected in textbooks and in the pedagogical literature, namely, the mechanical momentum that is carried by waves in linear theory of the wave equation.

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