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On the momentum of mechanical plane waves

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Abstract

Momentum of a mechanical, harmonic plane wave is derived and explained as a relativistic effect arising from the presence of tension in moving elements of the medium. Neglect of the relativistic corrections leads to the paradox, which is formulated and explained. Explicit results for momentum density resulting from tension for transverse and longitudinal waves are discussed. The idea of experiments for quantitative measurements of the momentum density is presented.

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Keywords: Theory of elasticity; Hidden momentum; Plane waves; Special relativity

1. Introduction

Momentum density $\mathrm{d}\boldsymbol{p}/\mathrm{d}V$ is strictly related to the energy flowing per unit surface per unit time [1]:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}V} = \frac{1}{c^2}\boldsymbol{S} \tag{1}$$

where c is the speed of light, and can be obtained from the Lorentz transformation of momentum fourvector. In case of electromagnetic fields vector $\boldsymbol{S}$ coincides with the Poynting vector [2]. Although the relation between the energy flow and the momentum density (1) is strictly valid, precise recognition of the momentum density is sometimes not straightforward [3]. In the case of mechanical waves, in particular acoustic plane waves in elastic medium, the situation requires careful analysis. Indeed, in the canonical courses of physics [1,4,5] one cannot find discussion of the question of the momentum density of such a wave. In particular, to our knowledge one can hardly find an explicit derivation of the momentum density when a textbook example, masses connected by springs, which is introductory example for the concept of sound and phonons, is discussed. Theoretical considerations presented in Refs. [6–11] do not suggest any possibility of measuring of the momentum density of plane waves. Some published results for momentum density of plane waves disagree with Eq. (1).

Let us consider the textbook example of infinite chain of springs with stiffness constant k connected with point masses m [12]. The length of the springs at equilibrium is L. Each mass is numbered by n, and vibrates about its positions of equilibrium only. Position of the nth mass is $x_n(t)$. Deformation of the system $\Psi^{(n)}$ (see Fig. 1a) understood as a departure from the equilibrium, is described by

$$\Psi^{(n)} = x_n(t) - nL \tag{2}$$

Equation of motion is

$$\ddot{\Psi}^{(n)} = -k/m(2\Psi^{(n)} - \Psi^{(n-1)} - \Psi^{(n+1)}) \tag{3}$$

and a harmonic running wave is described by

$$\Psi^{(n)} = \Psi_0 \cos(nqL - \omega t + \varphi) \tag{4}$$

where q is the wave vector, φ constant phase and the dispersion relation, $\omega(q)$, is

$$\omega = 2\sqrt{k/m}\,\sin\frac{Lq}{2} \tag{5}$$

Eq. (4) describes a sinusoidal wave travelling from left to right. The system transmits energy. This can be visualized in *gedanken* experiment. Let us cut the system just before mass $n+1$ and attach a body that moves with a friction (see

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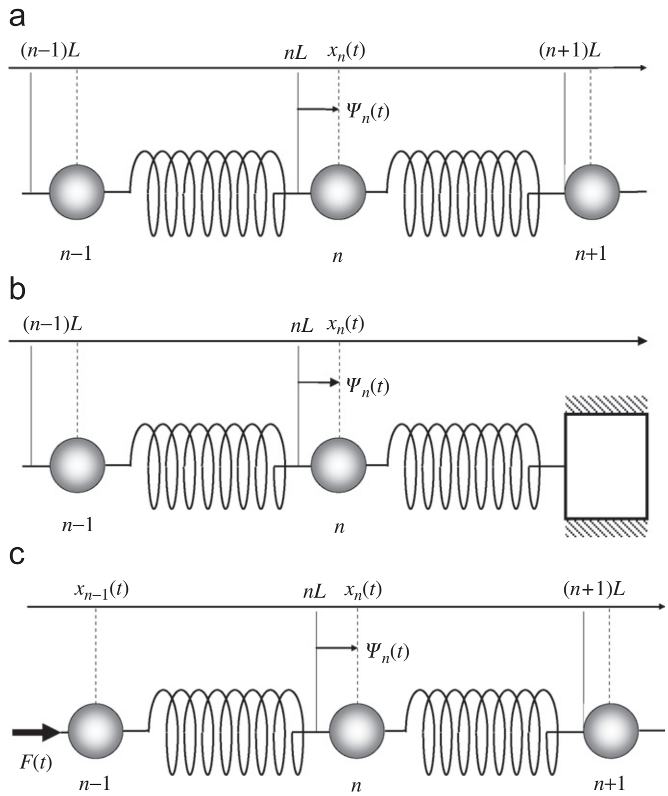

Fig. 1. (a) Part of infinite system of masses and springs, (b) transfer of energy to the receiver shown as box, (c) external force as a source of the plane wave. Solid vertical lines show equilibrium position of the balls.

Fig. 1b). We will observe dissipation of the wave energy by the body. In order to calculate the energy flow of the wave running from left to right per unit time, let us perform another *gedanken* experiment. Let us now cut the system before mass $n-1$ and apply an external force $F(t)$ such that all masses to the right will move according to Eq. (4) (see Fig. 1c). The work W per unit time of the force is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = F(t)\dot{\Psi}^{(n-1)} = -k(\Psi^{(n-2)} - \Psi^{(n-1)})\dot{\Psi}^{(n-1)} \tag{6}$$

The average power transmitted to the system is the energy flow:

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} \frac{\mathrm{d}W}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{2}\Psi_0^2 k\omega \sin Lq \tag{7}$$

where τ is the period of the vibrations $\tau = 2\pi/\omega$. In context of the generality of Eqs. (1), (7), should result in the momentum density of the particular example of mechanical plane wave (4):

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{1}{c^2 L}\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle = \frac{1}{c^2}\frac{\Psi_0^2 m\omega^2}{2L}\frac{\mathrm{d}\omega}{\mathrm{d}q} \tag{8}$$

The $\mathrm{d}\omega/\mathrm{d}q$ in Eq. (8) is the group velocity and can be obtained from dispersion relation (5). We do not write explicit result for $\mathrm{d}\omega/\mathrm{d}q$ in order to keep it as a group velocity.

The result (8) can be extended to three-dimensional body consisting of masses connected by springs forming simple cubic structure. For the longitudinal wave in the (1 0 0) direction we have the momentum density:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}V} = \frac{1}{c^2 L^3}\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle \frac{\boldsymbol{q}}{q} = \frac{1}{c^2}\frac{\Psi_0^2 \rho\omega^2}{2}\frac{\mathrm{d}\omega}{\mathrm{d}q}\frac{\boldsymbol{q}}{q} \tag{9}$$

The results (8) and (9) are simple consequences of Eq. (1). The right-hand sides of Eqs. (8) and (9) are the energy density of the wave multiplied by the group velocity and divided by the square of speed of light. Let us recognize the specific form of the expressions describing the momentum density in Eqs. (8) and (9). Mechanical momentum of the nth ball is just $m\dot{\Psi}^{(n)}$. However this quantity averaged over time yields zero. Also this quantity for certain time (summed over entire system) is zero. If one treats the system as relativistic one and corrects the momentum by a relativistic factor plus a relativistic contribution coming from potential energy of the moving spring, one also gets zero momentum density. This result can serve as paradox in context of Eqs. (8) and (9). In the next paragraphs we show that the "hidden momentum" results from the tension present in the moving springs.

2. Recognition of the "hidden momentum"

Let us consider in the inertial system O an elastic medium of density ρ at the rest. Consider next an element of the same medium in which an uniaxial stress is present, see Fig. 2. The energy of the compressed element is larger by potential energy resulting from the Hooks' law. Although it is obvious, we quote that none of the forces shown in Fig. 2 is performing the work in the O system.

Let us observe the elements of the bodies from the inertial system O', which is moving with respect to the O with velocity υ. We will observe Lorentz contraction of both elements and their constant velocity movement. There is, however, remarkable difference between the energy flows in both elements. In the uncompressed element there is a flow of mass only while in the compressed one, forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are performing a work, because these forces are acting on the ends moving with velocity $-\upsilon$. So there is an additional flow of the energy through the compressed element, and according to Eq. (1) an additional momentum density should be present.

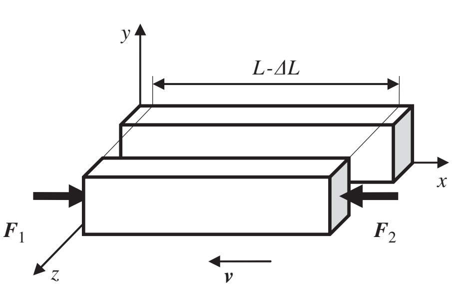

Fig. 2. Momentum density arising from stress and movement of the medium.

Quantitative transformation between tension and momentum density is given by so-called relativistic stress tensor $\boldsymbol{T}$ [4,6]. Its Cartesian components are:

$$\boldsymbol{T} = \begin{pmatrix} T_{00} & T_{0x} \\ T_{0x} & T_{xx} \end{pmatrix} \tag{10}$$

where T_{00} is the energy density, T_{xx} is the tension along x direction, T_{0x}/c is the x component of the momentum density. The y and z components of the $\boldsymbol{T}$ tensor (10) were neglected for simplicity.

In the O system the relativistic stress tensor of the compressed element in Fig. 2 is diagonal:

$$\boldsymbol{T} = \begin{pmatrix} u & 0 \\ 0 & \sigma \end{pmatrix} \tag{11}$$

where u is the relativistic energy density in the O system, being the sum of the mass-equivalent energy plus elastic potential energy, and σ is the tension along x direction. In the O' system components are obtained by Lorentz transformation $\boldsymbol{A}(\beta)$:

$$\boldsymbol{T}' = \tilde{\boldsymbol{A}}(-\beta)\cdot\boldsymbol{T}\cdot\boldsymbol{A}(-\beta) = \frac{1}{1-\beta^2}\begin{pmatrix} u+\sigma\beta^2 & \beta(u+\sigma) \\ \beta(u+\sigma) & \sigma+u\beta^2 \end{pmatrix} \tag{12}$$

where

$$\boldsymbol{A}(\beta) = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}, \quad \beta = \frac{\upsilon}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{13}$$

We see that stress s present in the element at rest results in extra momentum density when this element is moving with velocity υ. The total momentum density is given by the nondiagonal component of the $\boldsymbol{T}'$:

$$\frac{\mathrm{d}p_x}{\mathrm{d}V} = \frac{(u+\sigma)}{c^2(1-\beta^2)}\upsilon \tag{14}$$

3. Explicit derivation of the momentum density of the springs and masses

With the help of result (14) we are ready for direct determination of the momentum of the elastic wave (4). For the spring connecting nth and $(n+1)$st balls the momentum density is

$$\frac{p_n(t)}{L} = \frac{1}{c(1-\beta^2)}\left(\frac{k}{2L}(\Psi^{(n+1)}-\Psi^{(n)})^2 + k(\Psi^{(n+1)}-\Psi^{(n)})\right)\beta$$
$$\beta = \frac{\dot{\Psi}^{(n+1)}+\dot{\Psi}^{(n)}}{2c} \tag{15}$$

where in Eq. (15) velocity $\upsilon = \beta c$ was approximated by the velocity of the centre of the spring. The first term in the parenthesis, potential energy, multiplied by velocity and averaged over time yields zero, which was mentioned already at the end of second section. The second term, however, is proportional to the energy flow. Neglecting higher-order terms than $1/c^2$ in Eq. (15), we obtain after averaging:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\langle p_n(t)\rangle}{L} = \frac{1}{c^2}\frac{\Psi_0^2 m\omega^2}{2L}\frac{\mathrm{d}\omega}{\mathrm{d}q} + O\left(\frac{\dot{\Psi}^2}{c^2}\right) \tag{16}$$

which agrees with Eq. (8) and coincides with general rule (1). Eqs. (16) and (8) apply to the systems where no mass transport is present. Thus our results do not apply to systems of fluids where average mass transfer is present [8], or to the anharmonic systems where average position of the medium particle depends on the amplitude [10]. In two specific cases of waves with frequency tending to zero and maximal frequency the momentum density in Eq. (16) tend to zero. The maximal momentum density of the wave (3) is for $q = 2/L\arccos(3^{-1/2})$. Another comment is that Eq. (16) is valid only for travelling wave. In case of standing wave time average of Eq. (15) yields zero.

4. Momentum density of the acoustic plane waves in solids

In this paragraph we consider acoustic waves propagating in isotropic elastic medium. $\Psi(t,\boldsymbol{r})$ describes departure from the equilibrium. In special case of harmonic running wave

$$\Psi(t,\boldsymbol{r}) = \Psi_0\cos(\boldsymbol{qr}-\omega t-\varphi) \tag{17}$$

There are two types of plane waves: longitudinal and transverse. Introducing sound velocities c_L and c_T for the longitudinal and transverse waves, respectively [4]:

$$c_\mathrm{L} = \omega_\mathrm{L}/q_\mathrm{L} = \sqrt{\frac{(1-v)Y}{\rho(1+v)(1-2v)}}$$
$$c_\mathrm{T} = \omega_\mathrm{T}/q_\mathrm{T} = \sqrt{\frac{Y}{2\rho(1+v)}} \tag{18}$$

where Y is Young modulus, v the Poisson ratio and ρ the density, the energy flow in the two special cases can be written as

$$\boldsymbol{S} = \frac{1}{2}\Psi_0^2\rho\omega^2 c_i\frac{\boldsymbol{q}}{q},\ i = \mathrm{L,T} \tag{19}$$

Momentum density of acoustic wave can be immediately obtained from the energy flow and Eq. (1). Thus, the momentum density of the longitudinal and transverse plane wave should be

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}V} = \frac{1}{c^2}\boldsymbol{S} = \frac{1}{2c^2}\Psi_0^2\rho\omega^2 c_i\frac{\boldsymbol{q}}{q}, \quad i = \mathrm{L,T} \tag{20}$$

The result (20) can be obtained explicitly from the relativistic stress tensor. Let us consider first the stress induced by the wave. The strain tensor [4]:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial\Psi_i}{\partial x_j}+\frac{\partial\Psi_j}{\partial x_i}\right) \tag{21}$$

The stress tensor

$$\sigma_{ij} = \frac{Y}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}e_{kk}\right) \tag{22}$$

Without loosing generality, let us consider longitudinal wave with $\boldsymbol{q}$ vector parallel to the x -axis, see Fig. 3a. The relativistic stress tensor in the system moving together with the element of the medium

$$\boldsymbol{T} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & \sigma_{xx} & 0 & 0 \\ 0 & 0 & \sigma_{yy} & 0 \\ 0 & 0 & 0 & \sigma_{zz} \end{pmatrix} \tag{23}$$

The tensor in the rest frame has the form

$$\boldsymbol{T}' = \tilde{\boldsymbol{A}}(-\beta)\cdot\boldsymbol{T}\cdot\boldsymbol{A}(-\beta) \tag{24}$$

where

$$\boldsymbol{A}(\beta) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \frac{\dot{\Psi}}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{25}$$

The only nonzero element among $\boldsymbol{T}'_{0i}$ components of the $\boldsymbol{T}'$ tensor is

$$T'_{0x} = \frac{\dot{\Psi}}{c}(u+\sigma_{xx}) + O\left(\frac{\dot{\Psi}^2}{c^2}\right) \tag{26}$$

Time averaging of the product $\dot{\Psi}\cdot u$ yields zero. Thus from Eq. (26) we obtain

$$\langle T'_{0x}\rangle = \frac{1}{c}\langle\dot{\Psi}\sigma_{xx}\rangle = \frac{1}{2c}\Psi_0^2\rho\omega^2 c_{\mathrm{L}} + O\left(\frac{\dot{\Psi}^2}{c^2}\right) \tag{27}$$

We see that averaged T'_{0x} component Eq. (26) divided by c produces required momentum density, which agrees with Eq. (19) for longitudinal wave.

In similar way for the transverse wave propagating in y direction, and with polarization in x direction (see Fig. 3b), we find in the $\boldsymbol{T}$ tensor pure shear tension elements only:

$$\boldsymbol{T} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 0 & \sigma_{xy} & 0 \\ 0 & \sigma_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

After applying the Lorentz transformation (24) to (28), we arrive at the result

$$\boldsymbol{T}' = \begin{pmatrix} \gamma^2 u & \gamma^2 u\beta & \gamma\sigma_{xy}\beta & 0 \\ \gamma^2 u\beta & \gamma^2 u\beta^2 & \gamma\sigma_{xy} & 0 \\ \gamma\sigma_{xy}\beta & \gamma\sigma_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

We see that now there are two nonvanishing components of the momentum density in Eq. (28). However the time average $\langle T'_{0x}\rangle = 0$ and

$$\langle T'_{0y}\rangle = \frac{1}{2c}\Psi_0^2\rho\omega^2 c_{\mathrm{T}} + O\left(\frac{\dot{\Psi}^2}{c^2}\right) \tag{30}$$

We see that Eq. (30) divided by c produces required momentum density, which agrees with Eq. (19) for transverse wave.

5. Discussion of the "hidden momentum"

For deeper understanding of the origin of the "hidden momentum", let us consider two boxes of the same size (see Fig. 4). The upper box has mirrors at vertical ends located at distance L. Between the mirrors n monochromatic photons are located forming a uniform density of energy $u = E/V$. Volume of the second box is filled uniformly by the mass $m = E/c^2$.

Both boxes contain the same energy density u when observed in the system at rest. The situation is different when they are observed from the O' system. Because of Doppler effect photon energies are

$$E'_{\pm} = E\frac{1\pm\beta}{\sqrt{1-\beta^2}} \tag{31}$$

Two signs $\pm$ correspond to the photon momentum parallel or antiparallel to the velocity of O with respect to O'.

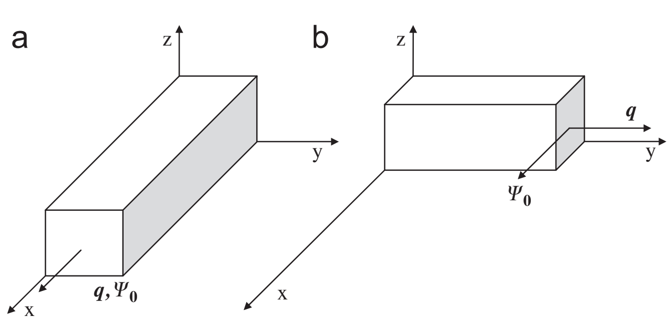

Fig. 3. Orientation of q and Ψ_0 vectors for (a) longitudinal and (b) transverse wave in elastic medium.

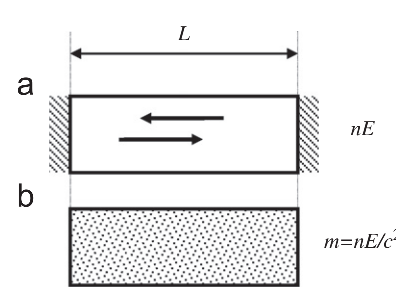

Fig. 4. (a) Box with mirror ends containing n photons and (b) uniformly filled with mass.

The photons travel between the mirrors within time

$$t'_{\pm} = t\sqrt{\frac{1 \pm \beta}{1 \mp \beta}} \tag{32}$$

where $t = L/c$. Total energy density and momentum density observed in the moving box containing photons are

$$u' = n\frac{E_+ t_+ + E_- t_-}{V'(t_+ + t_-)} = u\frac{1+\beta^2}{1-\beta^2} \tag{33}$$

$$\frac{dp'}{dV'} = n/c\frac{E_+ t_+ - E_- t_-}{V'(t_+ + t_-)} = u/c\frac{2\beta}{1-\beta^2} \tag{34}$$

respectively. The energy density and momentum density observed in the box containing dispersed mass are

$$u' = \frac{mc^2}{V'\sqrt{1-\beta^2}} = u\frac{1}{1-\beta^2} \tag{35}$$

$$\frac{dp'}{dV'} = \frac{m\beta}{V'\sqrt{1-\beta^2}} = u/c\frac{\beta}{1-\beta^2} \tag{36}$$

respectively. We see from Eqs. (33) and (35) that the box with photons has larger energy density in the O′ system by $(1+\beta^2)$, while from Eqs. (34) and (36) it follows that the box with photons has momentum density twice as large as box containing the mass only. This extra energy and momentum density result from the tension, which is present in the volume between mirrors, see disussion in Refs. [13–15]. Our straightforward estimations (31)–(36) can be formally performed after writing relativistic stress tensors in the O system. Diagonal elements are $T_{00}=T_{xx}=u$ for the upper box and $T_{00}=u$, $T_{xx}=0$ for the bottom box. Transforming components of these tensors to the moving frame with the use of Eq. (12) we arrive at the results (33)–(36).

The example with moving boxes shows that the tension contributes to the energy and momentum density and cannot be neglected. Result (14) shows, that macroscopic classical tension introduces corrections of the order of $1/c^2$ to the energy and momentum density of the moving bodies. Our considerations are closely related to the problem of moving charged condenser, for which similar paradox was indicated and explained [16], see also discussions in Refs. [13,15,17].

In similar way one can argue that shear stress in the moving body results in nonzero momentum density. To see clear example of this consider two systems A and B connected by elastic rods. In system A we apply the torque and perform rotation of the rods as shown in Fig. 5. In the system B the opposite torque is applied. This may be realized by dissipation of energy to B. Thus we transfer the energy from A to B. According to Eq. (1), the momentum density in vertical direction should be present in the rods. However, there is no vertical component of the velocity. Again, to explain this paradox one has to consider the relativistic stress tensor. The momentum density results from the shear stress and horizontal movement shown in Fig. 5. Formal derivation was presented in the paragraph 5 where momentum of transverse wave was estimated.

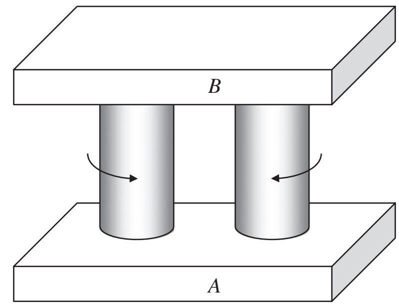

Fig. 5. Transfer of energy under shear stress. Two rods are used to prevent torque between A and B.

We should like to comment the results (8), (9) and (21) in the context of pseudomomentum of phonons. Pseudomomentum of a phonon, $\hbar\boldsymbol{q}$, is conserved in microscopical processes of scattering and this is understood as process of diffraction of the particle on the moving diffraction frame, as explained beautifully in Ref. [18]. The plane wave in elastic medium can be considered as coherent state of phonons. Thus for such coherent state of phonons, Eq. (1) should apply and also results (8), (9) and (20). *Gedanken* experiment illustrating this property is presented in next paragraph. Thus for a phonon with psedomomentum $\hbar\boldsymbol{q}$ and energy $\hbar\omega$ there should be momentum arising from Eq. (1) equal to

$$\boldsymbol{p} = \frac{\hbar\omega}{c^2}\frac{d\omega}{dq}\frac{\boldsymbol{q}}{q} \tag{37}$$

The momentum of phonon (37) can be detected only in very precise experiments since it is smaller than momentum of photon of the same wavelength by the factor (speed of light/velocity of sound)$^2 \approx 10^{10}$. However, a coherent state of phonons, describing a sound wave in an elastic solid, gives some hope for experimental detection of the momentum density.

6. Possibility of experimental observation of the plane wave momentum density

Results (8), (9) and (20) for momentum density of plane waves indicate clearly that experimental observation is difficult since momentum density is relativistic correction of the second order.

In the first type of proposed experiment let us consider a torus made of elastic medium. The running wave should be introduced to the torus. This can be done by mounting two systems, A and B, which couple to the medium and may generate the wave, see Fig. 6a. The systems A and B should have possibility of controlling the phase shift. Thus, by proper adjusting of the phase, a running wave in a well-defined direction could be introduced to the system. The torus is mounted so to allow its rotation along z-axis in Fig. 6a. A running wave is generated in the torus. It should

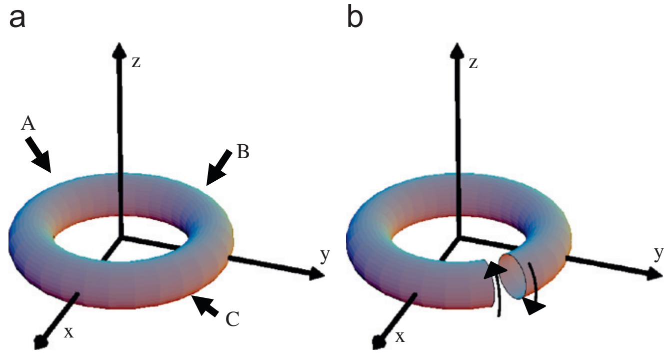

Fig. 6. (a) Experiment with running wave and (b) experiment with twisted tube.

be proved, that the introduced wave is a running wave and not a standing one. At certain point C the angular velocity of the torus must be observed. Shortly after A and B generators are switched off, the point C should start vibration around its initial position in the xy plane. Further, the running wave is dissipated and finally vibrations disappear and the angular momentum is transferred to the torus resulting in its steady rotation along z-axis.

Do we have any chance to observe it? Let us consider Al torus and introduce a wave with the amplitude of $\psi_0 = 0.1\,\text{mm}$. Assume arbitrarily that the length of the wave $\lambda = 1\,\text{cm}$. Under these conditions, the estimated linear velocity of the C point after dissipation of vibrations is

$$\upsilon = 2\pi^2 \frac{\Psi_0^2 c_s^3}{c^2 \lambda^3} = 3 \times 10^{-7}\ \text{m/s} \tag{38}$$

where assumed velocity of sound was $c_s = 5000\,\text{m/s}$. The experiment would be challenging, because initial maximal velocity of vibrations is 9 orders of magnitude larger than the final constant drift.

In the second type of experiment we propose measurement of angular momentum density resulting from the shear and movement. Basic idea is shown in Fig. 5. Quantitative measurements have to be performed in isolated system. Consider a tube of elastic medium bended to form a torus (see Fig. 6b). Two ends of the bended tube have to be twisted (see pair of arrows in Fig. 6b) and connected again. Thus there will be continuous shear in the bended tube forming a torus. Next, the bended tube should be rotated along its axis of symmetry, as shown for example by left curved arrow in Fig. 6b. During this rotation momentum density will locally appear along the tube and the whole torus will posses the angular momentum. After dissipation of energy of the rotation along axis of the tube, we will observe rotation of the whole torus along z-axis.

Both experiments have to perform in vacuum. The reason for this is the interaction of the travelling wave with surrounding gas and transfer of the angular momentum to the torus. Finally, it is worth to emphasise that both experiments are designed so, that an angular momentum will be measured and not a linear momentum. This is kind of approximation: the larger size of the torus with respect of the diameter of elastic rod (tube), the better approximation.

7. Conclusions

In order to explain behaviour of as simple system as springs and balls one has to consider momentum arising from tension in the moving springs. The same arguments apply to the system shown in Fig. 5 and to acoustic waves in the elastic medium. The momentum density, which follows from Eq. (1), should apply to any system, in particular to fluids. However, the mass transport should be carefully taken into account. For the presented derivations the mass transfer was assumed to be zero since we were dealing with elastic media. Concepts of pseudomomentum and relativistic momentum of phonons do not contradict each other because these two quantities are completely different. Coherent state of phonons allows experimental verification of the momentum of plane wave.

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