Department of physics, NTNU
TFY4340 Mesoscopic Physics
Spring 2010

Exercise 7

Assistance: Tuesday March 16.

a) Show that the Schrödinger equation (SE), $H\phi = E\phi$, for a 1D harmonic oscillator,

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega_0^2 y^2,$$

can be written in the form

$$\left[-\frac{d^2}{d\xi^2} + \xi^2\right]\phi(\xi) = \varepsilon\phi(\xi),$$

by introduction of dimensionless variables $\xi = y\sqrt{m\omega_0/\hbar}$ and $\varepsilon = 2E/\hbar\omega_0$.

b) Consider a 2DEG subject to a uniform perpendicular magnetic field $\boldsymbol{B} = B\hat{z}$ and a harmonic confining potential $V(y) = m\omega_0^2 y^2/2$. Use the Landau gauge $\boldsymbol{A} = -yB\hat{x}$, introduce the ansatz

$$\psi_{n,k}(x,y) = e^{ikx}\,\phi_n(y),$$

and show that the resulting equation for ϕ_n becomes

$$\left[-\frac{d^2}{d\eta^2} + (\eta-\kappa)^2 + \alpha^2\eta^2\right]\phi_n(\eta) = \varepsilon\phi_n(\eta)$$

in the dimensionless variables

$$\eta = y/l_B \;\;;\;\; \kappa = kl_B \;\;;\;\; \alpha = \omega_0/\omega_c \;\;;\;\; \varepsilon = 2E/\hbar\omega_c.$$

Here, $l_B = \sqrt{\hbar/eB}$ is the magnetic length, and $\omega_c = eB/m$ is the cyclotron frequency.

c) Show that the equation for $\phi_n(\eta)$ can be written as a 1D harmonic oscillator equation, that the solutions in original variables are

$$\psi_{n,k}(x,y) = e^{ikx}\,\phi_n(y - kL_B^2),$$

with

$$L_B^2 = \frac{\omega_c^2}{\omega_c^2 + \omega_0^2}\,l_B^2,$$

and that the energy spectrum is

$$E_{n,k} = \hbar\Omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2k^2}{2M_B},$$

with $\Omega^2 = \omega_c^2 + \omega_0^2$ and $M_B = m\,\Omega^2/\omega_0^2$.

d) Find the (probability) current density

$$\boldsymbol{j} = \frac{\hbar}{m}\mathrm{Im}\,(\psi^* \nabla \psi) + \frac{e}{m}\boldsymbol{A}|\psi|^2$$

for the states $\psi_{n,k}$ found above. (Hint: One of the components of $\boldsymbol{j}$ is zero, the other component will be an expression that contains ϕ_n^2.)

e) Find the *total* (probability) current along the channel,

$$I_x = \int dy j_x(y).$$

(Hint: Take advantage of the symmetry properties of ϕ_n, and assume that the functions ϕ_n are normalized.) Check the two limiting cases

$$\begin{array}{ll} (1) & \omega_0 \to 0 \ , \ \ \omega_c \neq 0 \\ (2) & \omega_c \to 0 \ , \ \ \omega_0 \neq 0 \end{array}$$