Department of physics, NTNU
TFY4340 Mesoscopic Physics
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Solution to Exercise 10

a) Kirchhoff's voltage law for the left loop (counting potential changes, going clockwise):

$$-\frac{V}{2}+\frac{Q_1}{C_1}+\frac{Q_g}{C_g}-U=0. \qquad (1)$$

For the right loop (also going clockwise):

$$U-\frac{Q_g}{C_g}+\frac{Q_2}{C_2}-\frac{V}{2}=0. \qquad (2)$$

For the outer loop (also going clockwise):

$$-V+\frac{Q_1}{C_1}+\frac{Q_2}{C_2}=0. \qquad (3)$$

The last equation is nothing but the sum of the first two equations, and brings nothing new. However, we will use it below. The island charge is

$$q=Q_1-Q_2-Q_g. \qquad (4)$$

So, we have 3 independent equations for the 3 unknown charges Q_1, Q_2, and Q_g. Multiplication of (1) by C_1 and (2) by C_2 yields

$$Q_1-Q_2=(C_1+C_2)U+\frac{1}{2}(C_1-C_2)V-\frac{C_1+C_2}{C_g}Q_g \overset{(4)}{=} q+Q_g,$$

which may be solved for Q_g:

$$Q_g=\frac{C_g}{C_\Sigma}\left\{\frac{1}{2}\left(C_1-C_2\right)V+\left(C_1+C_2\right)U-q\right\}. \qquad (5)$$

Next, (4) + C_2 multiplied by (3) gives us

$$\frac{C_1+C_2}{C_1}Q_1=C_2V+q+Q_g,$$

whereby insertion of the result for Q_g and solution with respect to Q_1 yields

$$Q_1=\frac{C_1}{C_\Sigma}\left\{\left(C_2+\frac{1}{2}C_g\right)V+C_gU+q\right\}. \qquad (6)$$

Finally, (4) yields

$$Q_2=\frac{C_2}{C_\Sigma}\left\{\left(C_1+\frac{1}{2}C_g\right)V-C_gU-q\right\}. \qquad (7)$$

b) In this question, we will try to find out how many extra electrons n will be on the island for given gate and bias voltages U and $\pm V/2$, respectively. (And for a given set of capacitances, of course.) The strategy will be the same as the one used in the lectures: Determine the energy change dE of the system when the island charge changes by an amount dq. Integration of the resulting equation will give us $E(q)$, i.e., $E(n)$, and we may discuss whether tunneling of an additional electron into or out of the island is favorable or not.

The electrostatic energy stored in the three capacitors is

$$\begin{aligned} E_{\mathrm{el}} &= \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} + \frac{Q_g^2}{2C_g} \\ &= \ldots \text{tedious algebra} \ldots \\ &= \frac{q^2}{2C_\Sigma} + q - \text{independent terms} \end{aligned} \qquad (8)$$

Thus, the contribution from E_{el} to the total energy change dE, when $q \to q + dq$, is

$$dE_{\mathrm{el}} = \frac{q\,dq}{C_\Sigma}. \qquad (9)$$

To this, we must add the work done by the island on the voltage sources U and $\pm V/2$. Let us consider $dq = -e$ due to tunneling of an electron into the island through tunneling junction 1. First of all, there is a contribution $(-dq)(-V/2) = dqV/2$ since a charge $-dq$ must be transported to the voltage source $-V/2$ (from ground), in order to compensate for the charge dq that has tunneled through junction 1. After this tunneling event, equilibrium is restored by an adjustment of the capacitor charges (on the island side):

$$Q_1 \to Q_1 + dQ_1 \quad , \quad -Q_2 \to -(Q_2 + dQ_2) \quad , \quad -Q_g \to -(Q_g + dQ_g).$$

On the outer side of each capacitor, this is compensated by charges $-dQ_1$, $+dQ_2$, and $+dQ_g$ flowing from voltage sources to the capacitor plates. The resulting work is

$$dW_1 = -\frac{V}{2}\left(dQ_1 - dq\right) + U \cdot (-dQ_g) + \frac{V}{2} \cdot (-dQ_2)\,. \qquad (10)$$

From a) we have (with constant V and U)

$$dQ_1 = \frac{C_1}{C_\Sigma} dq \quad , \quad dQ_2 = -\frac{C_2}{C_\Sigma} dq \quad , \quad dQ_g = -\frac{C_g}{C_\Sigma} dq.$$

Hence, the total energy change is

$$\begin{aligned} dE_1 &= dE_{\mathrm{el}} + dW_1 \\ &= \frac{dq}{C_\Sigma}\left\{q + \left(C_2 + \frac{C_g}{2}\right)V + C_g U\right\} \end{aligned} \qquad (11)$$

Integration on both sides, and insertion of $q = -ne$, yields

$$E_1(n) = \frac{1}{2C_\Sigma}\left\{\left(C_2 + \frac{C_g}{2}\right)V + C_g U - ne\right\}^2, \qquad (12)$$

plus some n–independent terms.
This is the energy given in the text (as E_{eq}).

c) Next, we want to calculate the energy change resulting from one additional electron tunneling through either of the two junctions, onto or out of the island. We start by looking at tunneling through junction 1, and onto the island (i.e., $n \to n+1$). We will eventually look at the special case with $C_1 = C_2 = C_g = C$, but let us first do the derivations with arbitrary capacitances:

$$\begin{aligned} \Delta E_1^+ &= E_1(n+1) - E_1(n) \\ &= \ldots \\ &= \frac{e}{C_\Sigma}\left\{ne + \frac{e}{2} - \left(C_2 + \frac{C_g}{2}\right)V - C_g U\right\} \end{aligned} \tag{13}$$

Similarly, for tunneling of an electron through junction 1 out of the island:

$$\begin{aligned} \Delta E_1^- &= E_1(n-1) - E_1(n) \\ &= \ldots \\ &= \frac{e}{C_\Sigma}\left\{-ne + \frac{e}{2} + \left(C_2 + \frac{C_g}{2}\right)V + C_g U\right\} \end{aligned} \tag{14}$$

The condition for stability against tunneling – one or the other way – through junction 1 is therefore (by requiring $\Delta E_1^+ > 0$ and $\Delta E_1^- > 0$)

$$-\frac{e}{2} < \left(C_2 + \frac{C_g}{2}\right)V + C_g U - ne < \frac{e}{2}. \tag{15}$$

For tunneling through junction 2, we may repeat all these steps and finally arrive at the following condition for stability against tunneling – one or the other way – through junction 2:

$$-\frac{e}{2} < -\left(C_1 + \frac{C_g}{2}\right)V + C_g U - ne < \frac{e}{2}. \tag{16}$$

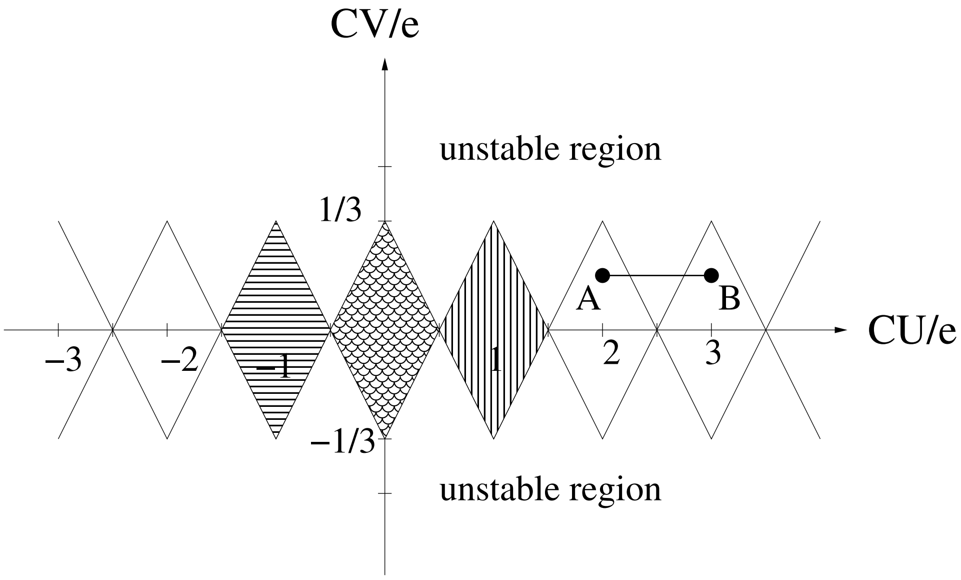
Now, we may specialize to equal capacitors, $C_1 = C_2 = C_g = C$, which means that $C_\Sigma = 3C$. The two stability criteria (15) and (16) then become

$$\begin{aligned} n - \frac{1}{2} &< \frac{(3V+2U)C}{2e} < n + \frac{1}{2} && (15') \\ n - \frac{1}{2} &< \frac{(-3V+2U)C}{2e} < n + \frac{1}{2} && (16') \end{aligned}$$

These represent 4 inequalities for a given value of n, e.g., for $n = 0$:

$$\begin{aligned} \frac{CV}{e} &> -\frac{2}{3}\frac{CU}{e} - \frac{1}{3} \\ \frac{CV}{e} &> \frac{2}{3}\frac{CU}{e} - \frac{1}{3} \\ \frac{CV}{e} &< -\frac{2}{3}\frac{CU}{e} + \frac{1}{3} \\ \frac{CV}{e} &< \frac{2}{3}\frac{CU}{e} + \frac{1}{3} \end{aligned}$$

Hence, the stable region for $n = 0$ in the CU/e – CV/e plane is a "diamond" centered at the origin, with corners in $(0, \pm 1/3)$ and $(\pm 1/2, 0)$. The full stability diagram becomes like this:



Here, the "fish shell" pattern is the $n = 0$ stability region, the vertically striped pattern is the $n = 1$ stability region, and the horizontally striped pattern is the $n = -1$ stability region.

Assume V is a positive bias, taking us (verically) to the level of the A–B line in the figure. This sign on V will tend to push electrons clockwise in our transistor circuit. An AC gate voltage $U(t) = U_0 + U_1 \cos(2\pi f t)$ with positive U_0 will represent oscillations along the line between A and B. Assume e.g. $U_1 > 0$. Then, increasing U from $U_0 - U_1$ (in A) to $U_0 + U_1$ (in B) allows for the tunneling of one electron onto the island ($n = 2 \rightarrow n = 3$), through junction 1. On the next half period, U is decreasing, from $U_0 + U_1$ in B to $U_0 - U_1$ in A. This results in tunneling of one electron out of the island ($n = 3 \rightarrow n = 2$), through junction 2. The net result is a DC current $I = ef$, proportional to the frequency of the gate voltage U.

Note that the accuracy in the current I will be the same as the accuracy in the frequency f. And frequencies can be created/measured with high accuracy! Thus, this single electron transistor can produce a current with high accuracy.