Department of physics, NTNU
TFY4340 Mesoscopic Physics
Spring 2010

Solution to Exercise 4

a) With the given gaussians, the inner product $\langle i|j\rangle$ is

$$\begin{aligned}\langle i|j\rangle &= C^2\int dx\, e^{-\kappa(x-ia)^2-\kappa(x-ja)^2}\\ &= C^2\int dx\, e^{-\kappa(2x^2-2(i+j)ax+(i^2+j^2)a^2)}\\ &= C^2 e^{-\kappa a^2(i-j)^2/2}\int e^{-2\kappa(x-(i+j)a/2)^2}\\ &= C^2 e^{-\kappa a^2(i-j)^2/2}\cdot\sqrt{\pi/2\kappa}\end{aligned}$$

where we have completed the square in the exponent, as suggested in the exercise. In this exercise, all integrals extend from $-\infty$ to $+\infty$. As long as $\kappa a^2/2 \gg 1$, and choosing $C = (2\kappa/\pi)^{1/4}$, we have

$$\langle i|j\rangle \simeq \delta_{ij}.$$

b) From the lectures, we have

$$\begin{aligned}\gamma &= -\langle 1|\sum_{m\neq 1} V_m|2\rangle\\ \varepsilon_0 &= \alpha-\alpha' = -\langle 2|\sum_{m\neq 2} V_m|2\rangle + \langle 1|\sum_{m\neq 1} V_m|1\rangle\end{aligned}$$

In the notation of the lectures (p 29), we have here chosen $j = l = 1$ in the matrix element α', $j = l = 2$ in the matrix element α, and $j = l - 1 = 1$ in the matrix element γ. Each of these matrix elements is a sum over the index m, and all terms are, at the outset, assumed to be "small". The largest of these small terms correspond to $m = 2$ in γ, $m = 2$ in α', and $m = 1$ and $m = 3$ in α. All these terms in α and α' are equal. Hence, if we keep only the largest term(s) in the sums over m, we end up with

$$\begin{aligned}\gamma &= -\langle 1|V_2|2\rangle\\ \varepsilon_0 &= -\langle 1|V_2|1\rangle\end{aligned}$$

Here, both matrix elements are integrals of products of gaussians. Just like in a), we may solve the integral by completing the square in the exponent. This yields

$$\begin{aligned}\gamma/C^2V_0 &= \sqrt{\frac{\pi}{2\kappa+\beta}}\exp\left(-a^2\cdot\frac{\kappa^2+\kappa\beta}{2\kappa+\beta}\right)\\ \varepsilon_0/C^2V_0 &= \sqrt{\frac{\pi}{2\kappa+\beta}}\exp\left(-a^2\cdot\frac{2\kappa\beta}{2\kappa+\beta}\right)\end{aligned}$$

and finally

$$y = \frac{\varepsilon_0}{\gamma} = \exp\left(a^2 \cdot \frac{\kappa(\kappa - \beta)}{2\kappa + \beta}\right),$$

which is larger than 1 if $\kappa > \beta$, i.e., if $|j\rangle$ decays faster than V_j.