Department of physics, NTNU
TFY4340 Mesoscopic Physics
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Solution to Exercise 7

a) Division of the SE with $\hbar\omega_0/2$ yields

$$\left[-\frac{\hbar}{m\omega_0}\frac{d^2}{dy^2}+\frac{m\omega_0}{\hbar}y^2\right]\phi=\varepsilon\phi.$$

Substitutions $y^2=\hbar\xi^2/m\omega_0$ and $dy^2=\hbar d\xi^2/m\omega_0$ turns the equation into the one given in the exercise.

b) Let's start by writing down the SE:

$$\left[-\frac{\hbar^2}{2m}\left(\nabla+\frac{ie\boldsymbol{A}}{\hbar}\right)^2+\frac{1}{2}m\omega_0^2y^2\right]\psi=E\psi.$$

With $\nabla=\hat{x}\partial/\partial x+\hat{y}\partial/\partial y$, $\boldsymbol{A}=-yB\hat{x}$, and $\psi=\exp(ikx)\phi_n(y)$, the SE becomes

$$\left[-\frac{\hbar^2}{2m}\left(ik-i\frac{eyB}{\hbar}\right)^2-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}+\frac{1}{2}m\omega_0^2y^2\right]\phi_n(y)=E\phi_n(y).$$

We note that

$$\frac{\hbar^2}{2m}\cdot\frac{1}{l_B^2}=\frac{\hbar^2}{2m}\cdot\frac{eB}{\hbar}=\frac{1}{2}\hbar\frac{eB}{m}=\frac{1}{2}\hbar\omega_c.$$

Thus, if we divide the SE with this factor, we obtain

$$\left[l_B^2\left(k-\frac{eyB}{\hbar}\right)^2-l_B^2\frac{\partial^2}{\partial y^2}+m^2\omega_0^2l_B^2y^2/\hbar^2\right]\phi_n(y)=\varepsilon\phi_n(y).$$

On the left hand side, the first term is $(\kappa-\eta)^2=(\eta-\kappa)^2$, since $\kappa=kl_B$ and $\eta=y/l_B=yl_B/l_B^2=yl_BeB/\hbar$. The second term is obviously $\partial^2/\partial\eta^2$. The third term is

$$m^2\omega_0^2l_B^4\eta^2/\hbar^2=m^2\omega_0^2\frac{\hbar^2}{e^2B^2}\eta^2/\hbar^2=\frac{\omega_0^2}{\omega_c^2}\eta^2=\alpha^2\eta^2.$$

Consequently, the equation for $\phi_n(\eta)$ is

$$\left[-\frac{d^2}{d\eta^2}+(\eta-\kappa)^2+\alpha^2\eta^2\right]\phi_n(\eta)=\varepsilon\phi_n(\eta).$$

c) We write the two terms inside the brackets as

$$\begin{aligned}(\eta-\kappa)^2+\alpha^2\eta^2 &= (1+\alpha^2)\eta^2-2\kappa\eta+\kappa^2\\ &= (1+\alpha^2)\left(\eta-\frac{\kappa}{1+\alpha^2}\right)^2+\kappa^2-\frac{\kappa^2}{1+\alpha^2}\\ &= (1+\alpha^2)\left(\eta-\frac{\kappa}{1+\alpha^2}\right)^2+\frac{\alpha^2\kappa^2}{1+\alpha^2}\end{aligned}$$

Hence, the equation for $\phi_n(\eta)$ can be written

$$\left[-\frac{d^2}{d\eta^2}+(1+\alpha^2)\left(\eta-\frac{\kappa}{1+\alpha^2}\right)^2\right]\phi_n(\eta)=\left(\varepsilon-\frac{\alpha^2\kappa^2}{1+\alpha^2}\right)\phi_n(\eta).$$

Comparison with the dimensionless equation in *a*) shows that this is an equation for a 1D harmonic oscillator, centered at $\eta_c=\kappa/(1+\alpha^2)$, and with a redefined energy eigenvalue $\varepsilon-\alpha^2\kappa^2/(1+\alpha^2)$. In addition, the factor $1+\alpha^2=1+\omega_0^2/\omega_c^2=\Omega^2/\omega_c^2$ in front of the quadratic term on the left hand side means that this equation comes from a harmonic oscillator equation where the potential energy is not $m\omega_c^2y^2/2$, but rather $m\Omega^2y^2/2$. Hence, we *know* that the dimensionless energy on the right hand side actually equals $(n+1/2)\hbar\Omega/(1/2)\hbar\omega_c$.
Now, we are ready to transform back to original variables. First, the center position of the harmonic oscillator wave functions:

$$\begin{aligned}y_c &= l_B\eta_c=l_B\kappa/(1+\alpha^2)\\ &= l_B^2k\omega_c^2/\Omega^2=L_B^2k\end{aligned}$$

Next, the energy spectrum:

$$\begin{aligned}(n+\frac{1}{2})\hbar\Omega &= \frac{1}{2}\hbar\omega_c\left(\varepsilon-\frac{\alpha^2\kappa^2}{1+\alpha^2}\right)\\ &= E-\frac{1}{2}\hbar\frac{eB}{m}\frac{\omega_0^2}{\Omega^2}k^2\frac{\hbar}{eB}\\ &= E-\frac{\hbar^2k^2}{2M_B}\end{aligned}$$

Therefore,

$$E=E_{n,k}=(n+\frac{1}{2})\hbar\Omega+\frac{\hbar^2k^2}{2M_B}.$$

The presence of the confining potential $V(y)$ has turned the Landau levels into "Landau bands": Electrons at the Fermi level distribute their energy E_F between discrete levels $(n+1/2)\hbar\Omega$ and a continuous "free particle like" part $\hbar^2k^2/2M_B$, suggesting electron transport along the 1D channel.

d) Since $\boldsymbol{A}$ has no y–component and the functions $\phi_n(y-L_B^2k)$ are real, we have $j_y=0$. The x–component, however, is not zero:

$$j_x = \frac{\hbar}{m}\mathrm{Im}\left(e^{-ikx}\phi_n\frac{\partial}{\partial x}e^{ikx}\phi_n\right)-\frac{eyB}{m}\phi_n^2$$

$$
\begin{aligned}
&= \left(\frac{\hbar k}{m} - \frac{eyB}{m}\right)\phi_n^2 \\
&= \omega_c\left(l_B^2 k - y\right)\phi_n^2(y - L_B^2 k).
\end{aligned}
$$

We see that $j_x > 0$ if $y < l_B^2 k$ and $j_x > 0$ if $y > l_B^2 k$.

e) The total probability current along the channel is

$$I_x = \int dy j_x(y) = \omega_c \int dy(l_B^2 k - y)\phi_n^2(y - L_B^2 k).$$

We take the hint given in the exercise and use the property

$$\phi_n^2(y - L_B^2 k) = \phi_n^2(L_B^2 k - y),$$

which must be true, since the harmonic oscillator functions have a certain parity, either $\phi_n(x) = \phi_n(-x)$ or $\phi_n(x) = -\phi_n(-x)$. Hence, if we rewrite I_x slightly,

$$I_x = \omega_c \int dy \left[-(y - L_B^2 k) - (L_B^2 - l_B^2)k\right]\phi_n^2(y - L_B^2 k),$$

we see that the first term gives an odd integrand which integrates to zero, whereas the second term is simply the constant $-\omega_c(L_B^2 - l_B^2)k$ multiplied by the normalization integral

$$\int dy \phi_n^2(y - L_B^2 k) = 1.$$

Hence, the total probability current is

$$I_x = \omega_c(l_B^2 - L_B^2)k = \ldots = \frac{\hbar k}{m}\frac{\omega_0^2}{\Omega^2}.$$

Limiting cases:

1. $I_x(\omega_0 \to 0) = 0$
2. $I_x(\omega_c \to 0) = \frac{\hbar k}{m}$

These limits are both quite reasonable: When $\omega_0 \to 0$, we have an infinite 2DEG, without any borders, and the electrons are simply spinning around without getting anywhere. In other words, $I_x = 0$. When $\omega_c \to 0$, the magnetic field is turned off, and we simply have electrons moving along the 1D channel with velocity $v_x = \hbar k/m$.