Department of physics, NTNU
TFY4340 Mesoscopic Physics
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Solution to Exercise 8

a) From the Büttiker–Landauer formula (first equation in the exercise), we have diagonal elements

$$\Gamma_{\alpha\alpha} = \frac{2e^2}{h} \sum_{\beta\neq\alpha} T_{\beta\alpha}$$

and off–diagonal elements

$$\Gamma_{\alpha\beta} = -\frac{2e^2}{h} T_{\alpha\beta}.$$

From the symmetry relation $T_{\alpha\beta}(B) = T_{\beta\alpha}(-B)$, we have directly for the off–diagonal matrix elements

$$\Gamma_{\alpha\beta}(B) = \Gamma_{\beta\alpha}(-B).$$

Combination of the equilibrium condition and Onsager symmetry yields

$$\sum_{\beta\neq\alpha} T_{\beta\alpha}(B) = \sum_{\beta\neq\alpha} T_{\alpha\beta}(B) = \sum_{\beta\neq\alpha} T_{\beta\alpha}(-B),$$

and hence

$$\Gamma_{\alpha\alpha}(B) = \Gamma_{\alpha\alpha}(-B).$$

b) We write out the equations (with $V_3 = 0$):

$$\begin{aligned}
I_1 &= \frac{2e^2}{h}\left\{\sum_{\beta\neq 1} T_{\beta 1} V_1 - T_{12} V_2 - T_{14} V_4\right\} \\
I_2 &= \frac{2e^2}{h}\left\{\sum_{\beta\neq 2} T_{\beta 2} V_2 - T_{21} V_1 - T_{24} V_4\right\} \\
I_3 &= -I_1 = \frac{2e^2}{h}\left\{-T_{31} V_1 - T_{32} V_2 - T_{34} V_4\right\} \\
I_4 &= -I_2 = \frac{2e^2}{h}\left\{\sum_{\beta\neq 4} T_{\beta 4} V_4 - T_{41} V_1 - T_{42} V_2\right\}
\end{aligned}$$

Pairwise summation of eqn 1 and 3, respectively eqn 2 and 4, yields

$$\begin{aligned}
0 &= (T_{21} + T_{41})V_1 - (T_{12} + T_{32})V_2 - (T_{14} + T_{34})V_4 \\
0 &= -(T_{21} + T_{41})V_1 + (T_{12} + T_{32})V_2 + (T_{14} + T_{34})V_4
\end{aligned}$$

which are identical equations. This shows that we had only 3, and not 4, independent equations for the 3 voltages V_1, V_2, and V_4 (for given currents I_1 and I_2).

Here, we need the currents in terms of the *difference* $d = V_2 - V_4$, whereas we are not interested in the sum $s = V_2 + V_4$. Therefore, we rewrite the last equation,

$$\begin{aligned} 0 &= -(T_{21} + T_{41})V_1 + (T_{12} + T_{32})(s + d)/2 + (T_{14} + T_{34})(s - d)/2 \\ &= -(T_{21} + T_{41})V_1 + (T_{12} + T_{32} + T_{14} + T_{34})s/2 + (T_{12} + T_{32} - T_{14} - T_{34})d/2 \end{aligned}$$

where we have used $V_2 = (s + d)/2$ and $V_4 = (s - d)/2$. From this equation, we can express s in terms of V_1, d, and $T_{\alpha\beta}$:

$$s = \frac{1}{\tau} \left\{ 2(T_{21} + T_{41})V_1 + (T_{14} + T_{34} - T_{12} - T_{32})d \right\},$$

with

$$\tau = T_{12} + T_{32} + T_{14} + T_{34}.$$

Clearly, τ will enter the elements of the matrix $\boldsymbol{\gamma}$, so let us study the symmetry properties of τ first. We may write (using $\sum_{\beta\neq\alpha} T_{\beta\alpha} = \sum_{\beta\neq\alpha} T_{\alpha\beta}$)

$$\begin{aligned} \tau &= \sum_{\beta\neq 2} T_{\beta 2} - T_{42} + \sum_{\beta\neq 4} T_{\beta 4} - T_{24} \\ &= \sum_{\beta\neq 2} T_{2\beta} - T_{42} + \sum_{\beta\neq 4} T_{4\beta} - T_{24} \\ &= T_{21} + T_{23} + T_{41} + T_{43}. \end{aligned}$$

These two expressions for τ, together with Onsager symmetry ($T_{\alpha\beta}(B) = T_{\beta\alpha}(-B)$), show that

$$\tau(B) = \tau(-B).$$

What remains now is to write the equations for I_1 and I_2 in terms of V_1, s, and d, and eliminate s with the expression found above.
We start with I_1:

$$\begin{aligned} I_1 &= \frac{2e^2}{h} \left\{ \sum_{\beta\neq 1} T_{\beta 1} V_1 - \frac{s}{2}(T_{12} + T_{14}) - \frac{d}{2}(T_{12} - T_{14}) \right\} \\ &= \frac{2e^2}{h} \left\{ \sum_{\beta\neq 1} T_{\beta 1} - \frac{(T_{21} + T_{41})(T_{12} + T_{14})}{\tau} \right\} V_1 \\ &+ \frac{2e^2}{h} \left\{ -\frac{T_{14} + T_{34} - T_{12} - T_{32}}{2\tau} \cdot (T_{12} + T_{14}) - \frac{1}{2}(T_{12} - T_{14}) \right\} d \end{aligned}$$

Hence, the $(1, 1)$–element is

$$\gamma_{11} = \frac{2e^2}{h\tau} \left\{ \tau \sum_{\beta\neq 1} T_{\beta 1} - (T_{21} + T_{41})(T_{12} + T_{14}) \right\},$$

and it is clear that

$$\gamma_{11}(B) = \gamma_{11}(-B).$$

The first sum is symmetric in B, and the second term is a product of two sums, where a change of sign in B turns one sum into the other, and vice versa.
The $(1,2)$–element is

$$\begin{aligned}
\gamma_{12} &= -\frac{e^2}{h\tau}\left\{(T_{14}+T_{34}-T_{12}-T_{32})(T_{12}+T_{14})+(T_{14}+T_{34}+T_{12}+T_{32})(T_{12}-T_{14})\right\} \\
&= -\frac{e^2}{h\tau}\left\{2(T_{14}+T_{34})T_{12}-2(T_{12}+T_{32})T_{14}\right\} \\
&= \frac{2e^2}{h\tau}\left\{T_{32}T_{14}-T_{34}T_{12}\right\}
\end{aligned}$$

Similarly, for I_2:

$$\begin{aligned}
I_2 &= \frac{2e^2}{h}\left\{\sum_{\beta\neq 2} T_{\beta 2}V_2 - T_{21}V_1 - T_{24}V_4\right\} \\
&= \frac{2e^2}{h}\left\{(T_{12}+T_{32}+T_{42})\cdot\frac{1}{2}(s+d) - T_{21}V_1 - T_{24}\cdot\frac{1}{2}(s-d)\right\} \\
&= \frac{2e^2}{h}\left\{-T_{21}+\frac{1}{\tau}(T_{21}+T_{41})(T_{12}+T_{32}+T_{42}-T_{24})\right\}V_1 \\
&+ \frac{2e^2}{h}\left\{\frac{1}{2\tau}(T_{12}+T_{32}+T_{42}-T_{24})(T_{14}+T_{34}-T_{12}-T_{32})+\frac{1}{2}(T_{12}+T_{32}+T_{42}+T_{24})\right\}d
\end{aligned}$$

Hence, the $(2,2)$–element is (where we use the expression for τ found earlier)

$$\begin{aligned}
\gamma_{22} &= \frac{e^2}{h\tau}\left\{(T_{12}+T_{32}+T_{42}-T_{24})(T_{14}+T_{34}-T_{12}-T_{32})\right\} \\
&+ \frac{e^2}{h\tau}\left\{(T_{12}+T_{32}+T_{42}+T_{24})(T_{14}+T_{34}+T_{12}+T_{32})\right\} \\
&= \frac{e^2}{h\tau}\left\{2(T_{12}+T_{32}+T_{42})(T_{14}+T_{34})+2T_{24}(T_{12}+T_{32})\right\} \\
&= \frac{2e^2}{h\tau}\left\{(T_{14}+T_{24}+T_{34})(T_{12}+T_{32})+T_{42}(T_{14}+T_{34})\right\} \\
&= \frac{2e^2}{h\tau}\left\{\tau\sum_{\beta\neq 4}T_{\beta 4}+T_{42}(T_{14}+T_{34})-(T_{14}+T_{34})(T_{14}+T_{24}+T_{34})\right\} \\
&= \frac{2e^2}{h\tau}\left\{\tau\sum_{\beta\neq 4}T_{\beta 4}-(T_{41}+T_{43})(T_{14}+T_{34})\right\}
\end{aligned}$$

This expression will be unchanged if we invert the direction of the magnetic field, hence $\gamma_{22}(B) = \gamma_{22}(-B)$.
Finally, the $(2,1)$–element is

$$\begin{aligned}
\gamma_{21} &= \frac{2e^2}{h\tau}\left\{-T_{21}(T_{12}+T_{32}+T_{14}+T_{34})+(T_{21}+T_{41})(T_{12}+T_{32}+T_{42}-T_{24})\right\} \\
&= \frac{2e^2}{h\tau}\left\{-T_{21}T_{14}-T_{21}T_{34}+T_{21}T_{42}-T_{21}T_{24}+T_{41}T_{12}+T_{41}T_{32}+T_{41}T_{42}-T_{41}T_{24}\right\} \\
&= \ldots = \frac{2e^2}{h\tau}\left\{T_{23}T_{41}-T_{43}T_{21}\right\}
\end{aligned}$$

Now, since $T_{\alpha\beta}(B) = T_{\beta\alpha}(-B)$, and $\tau(B) = \tau(-B)$, we see that

$$\gamma_{12}(B) = \gamma_{21}(-B).$$

In conclusion, the 2×2 matrix $\boldsymbol{\gamma}$ has the property

$$\boldsymbol{\gamma}(B) = \boldsymbol{\gamma}^T(-B).$$

c) The condition $I_2 = 0$ yields

$$V_1 = -\frac{\gamma_{22}}{\gamma_{21}} \left(V_2 - V_4\right),$$

and

$$I_1 = \left(-\gamma_{11}\frac{\gamma_{22}}{\gamma_{21}} + \gamma_{12}\right) \left(V_2 - V_4\right),$$

i.e.,

$$R_{13,24} = -\frac{\gamma_{21}}{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}.$$

Similarly, the condition $I_1 = 0$ yields

$$V_2 - V_4 = -\frac{\gamma_{11}}{\gamma_{12}} V_1,$$

and

$$I_2 = \left(\gamma_{21} - \frac{\gamma_{11}}{\gamma_{12}}\gamma_{22}\right) V_1,$$

i.e.,

$$R_{24,13} = -\frac{\gamma_{12}}{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}.$$

Since $\gamma_{12}(-B) = \gamma_{21}(B)$, we see immediately that

$$R_{13,24}(B) = R_{24,13}(-B).$$

Note that these 4–point resistances may have both signs. This is nothing more strange than the fact that Hall voltages may have both signs. Of course, the *dissipation* is always positive.