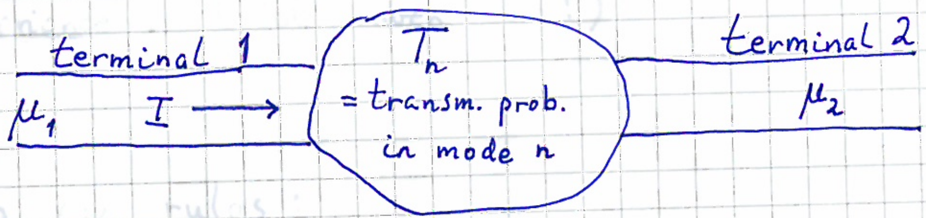
$\boxed{22.03.10}$

Büttiker - Landauer formalism

Recap., Landauer formula

(= "Two - terminal linear response formula") :



$$\mu_1 - \mu_2 = -e\,(V_1 - V_2) = -e \cdot \Delta V$$

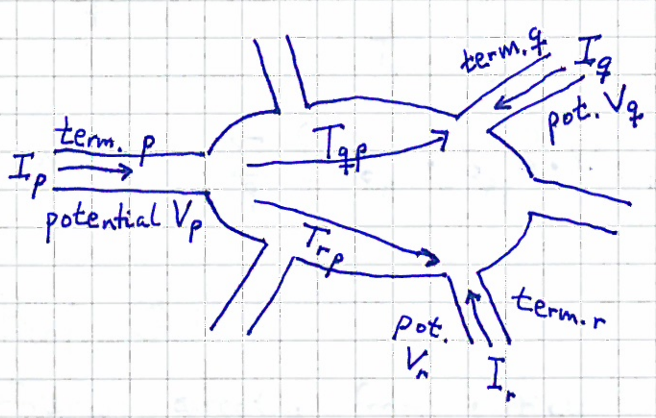
$$I = \sum_n I_n = \left\{ \frac{2e^2}{h} \sum_n T_n(E_F) \right\} \cdot \Delta V = \frac{2e^2}{h}\, T(E_F) \cdot \Delta V$$

$$= \left\{ \sum_n G_n \right\} \cdot \Delta V = G \cdot \Delta V$$

May also write: $I_1 = \frac{2e^2}{h} \left\{ T_{2\leftarrow 1} \cdot V_1 - T_{1\leftarrow 2} \cdot V_2 \right\}$

with $T_{i\leftarrow j} \equiv T_{ij} = \sum_n T_n(E_F)$ = total transm. "sum" from term. j to i

Multi-terminal generalization by M. Büttiker (~1985):



M-term. device (here: M=6)

$T_{qp} = T_{q\leftarrow p}$ = total transm. sum from p to q

$$I_p = \left(-\frac{2e}{h}\right)\sum_q \left\{ T_{qp}\cdot\mu_p - T_{pq}\cdot\mu_q \right\}$$

$$= \frac{2e^2}{h}\sum_q \left\{ T_{qp} V_p - T_{pq} V_q \right\}$$

[NB: $\sum_q$ means $\sum_{q\neq p}$]

$$\Rightarrow \boxed{I_p = \sum_q \left\{ G_{qp} V_p - G_{pq} V_q \right\}}$$ Büttiker formula

Onsager symmetry relation:

$$G_{qp}(-\vec{B}) = G_{pq}(\vec{B})$$ $[\Rightarrow G_{qp} = G_{pq}$ if $\vec{B}=0]$

Follows from particle conservation and time reversal invariance. (Not proven here!)

Two sum rules:

(1) $\sum_q G_{qp} = \sum_q G_{pq}$

(2) $\sum_p I_p = 0$

Proof, 1:

In equilibrium: $V_1 = V_2 = \dots = V_M = V$

$I_1 = I_2 = \dots = I_M = 0$

$$\Rightarrow 0 = \sum_q \{G_{qp} - G_{pq}\} \cdot V \quad \Rightarrow \sum_q G_{qp} = \sum_q G_{pq} \qquad \underline{\text{qed}}$$

Proof, 2:

Follows directly from particle conservation!

Is also built into the formalism, since

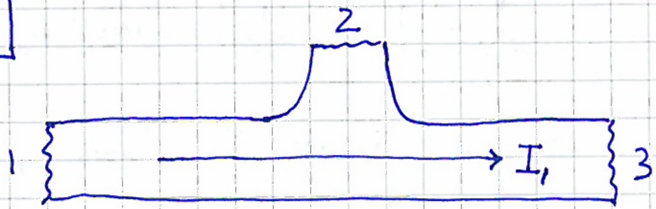
$$\sum_p \sum_{q \neq p} \{G_{qp} V_p - G_{pq} V_q\} = 0 \quad \text{(trivially)} \qquad \underline{\text{qed}}$$

Simplification via 1:

$$\boxed{I_p = \sum_q G_{pq} (V_p - V_q)} \quad \text{Büttiker formula}$$

Note: Only $M-1$ <u>independent</u> equations, since $\sum_p I_p = 0$

Example:



Find $R_{13,23} = \{(V_2 - V_3)/I_1\}_{I_2 = 0}$

[Notation: $R_{\alpha\beta,\gamma\delta} = \dfrac{\text{voltage between term. } \gamma, \delta}{\text{current from term. } \alpha \text{ to term. } \beta}$]

<u>Solution:</u>

From Büttiker:
$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} G_{12} + G_{13} & -G_{12} & -G_{13} \\ -G_{21} & G_{21} + G_{23} & -G_{23} \\ -G_{31} & -G_{32} & G_{31} + G_{32} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Only potential differences have physical relevance

$\Rightarrow$ we choose (e.g.) $V_3 = 0$

Also: $I_3 = -I_1 - I_2$ $\Rightarrow$ 3rd eqn. may be dropped

$$\Rightarrow \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} G_{12} + G_{13} & -G_{12} \\ -G_{21} & G_{21} + G_{23} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Here: $I_2 = 0$

$$\Rightarrow \quad I_1 = (G_{12} + G_{13}) V_1 - G_{12} V_2$$
$$0 = -G_{21} V_1 + (G_{21} + G_{23}) V_2$$

$$\Rightarrow I_1 = \left\{ (G_{12} + G_{13}) \cdot \frac{G_{21} + G_{23}}{G_{21}} - G_{12} \right\} V_2$$

$$\Rightarrow R_{13,23} = \frac{V_2}{I_1} = \underline{\underline{\frac{G_{21}}{G_{13} G_{21} + G_{12} G_{23} + G_{13} G_{23}}}}$$

[Find $R_{23,13} = \left(\frac{V_1}{I_2} \right)_{I_1 = 0}$!]

The Quantum Hall Effect

[First experiment: K. von Klitzing et al, 1980]

Edge states:

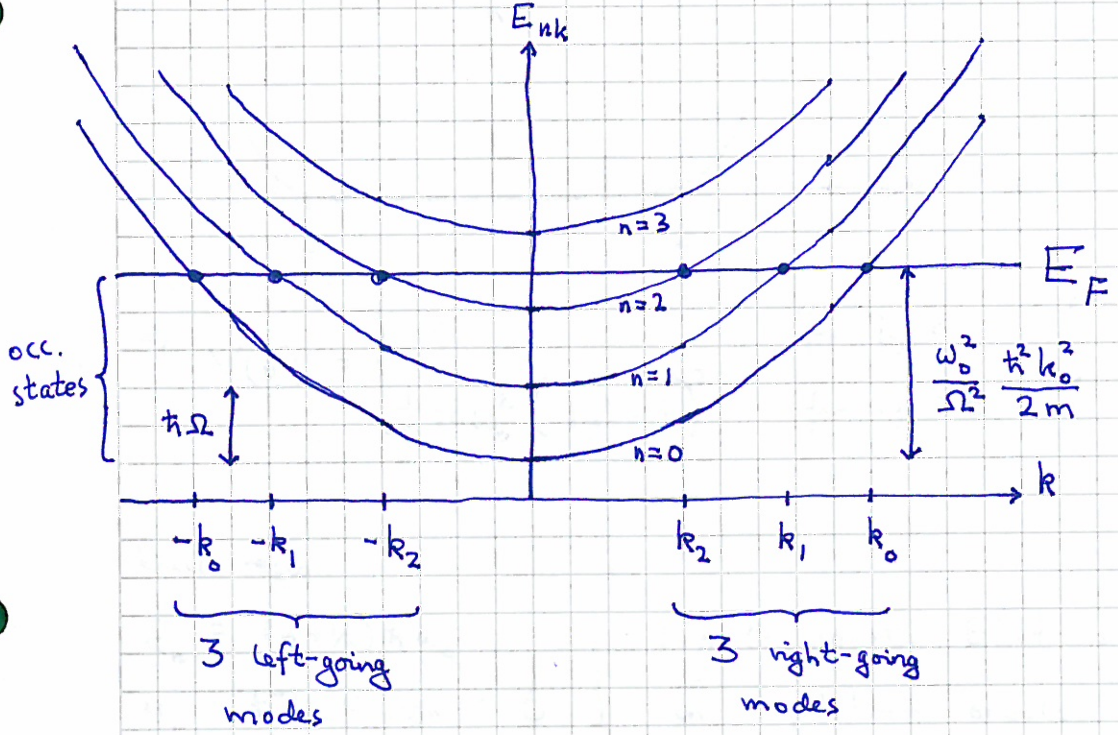
From Exercise 7:

$V(y) = \frac{1}{2} m \omega_0^2 y^2$, $\vec{A} = (-yB, 0, 0)$ $[\Rightarrow \vec{B} = \nabla \times \vec{A} = B\hat{z}]$

$\Rightarrow \psi_{nk} = e^{ikx} \phi_n \left(\frac{y - y_{nk}}{\ell_B} \right)$; $E_{nk} = \hbar\Omega(n + \frac{1}{2}) + \frac{\omega_0^2}{\Omega^2} \frac{\hbar^2 k^2}{2m}$

$$y_{nk} = k \cdot L_B^2 = k \cdot \frac{\omega_c^2}{\Omega^2} \ell_B^2 = k \cdot \frac{\omega_c^2}{\omega_c^2 + \omega_0^2} \cdot \frac{\hbar}{eB}$$

Transport takes place at $E_{nk} = E_F$:



$\Rightarrow$ This is similar to the 1D "waveguide":

1D subbands with DOS pr unit length

$$\rho_n^+(E) = \rho_n^-(E) = 2/h\cdot v_n(E)$$

$$\Rightarrow \rho_n^\pm(E)\cdot v_n(E) = \frac{2}{h} = \text{const.}$$

$\Rightarrow$ Landauer/Büttiker formulas are OK!

Transverse location:

$$y_{nk} = k_n(E_F)\cdot L_B^2 = \pm\left\{\frac{2m\Omega^2}{\hbar^2\omega_0^2}\left[E_F - (n+1/2)\hbar\Omega\right]\right\}^{1/2}\cdot L_B^2$$

if $k>0$, $y_{nk}>0$ (and $y_{0k} > y_{1k} > y_{2k} > \dots$)

if $k<0$, $y_{nk}<0$

Since $\Phi_n((y-y_{nk})/\ell_B) \sim \exp\left\{-\frac{1}{2}\left(\frac{y-y_{nk}}{\ell_B}\right)^2\right\}$, these states are <u>exponentially localized</u> around $y=y_{nk}$, on length scale $\Delta y \sim \ell_B = \sqrt{\hbar/eB} \approx 30\text{ nm}/\sqrt{B}$ (with B in tesla)

More typical confining potential:

$$\Rightarrow \quad V(y) = \begin{cases} 0 & \text{for } |y| < b \quad (\text{"bulk region"}) \\ \frac{1}{2} m\omega_0^2 \left(|y| - b\right)^2 & \text{for } |y| > b \quad (\text{"edge region"}) \end{cases}$$

Bulk region:

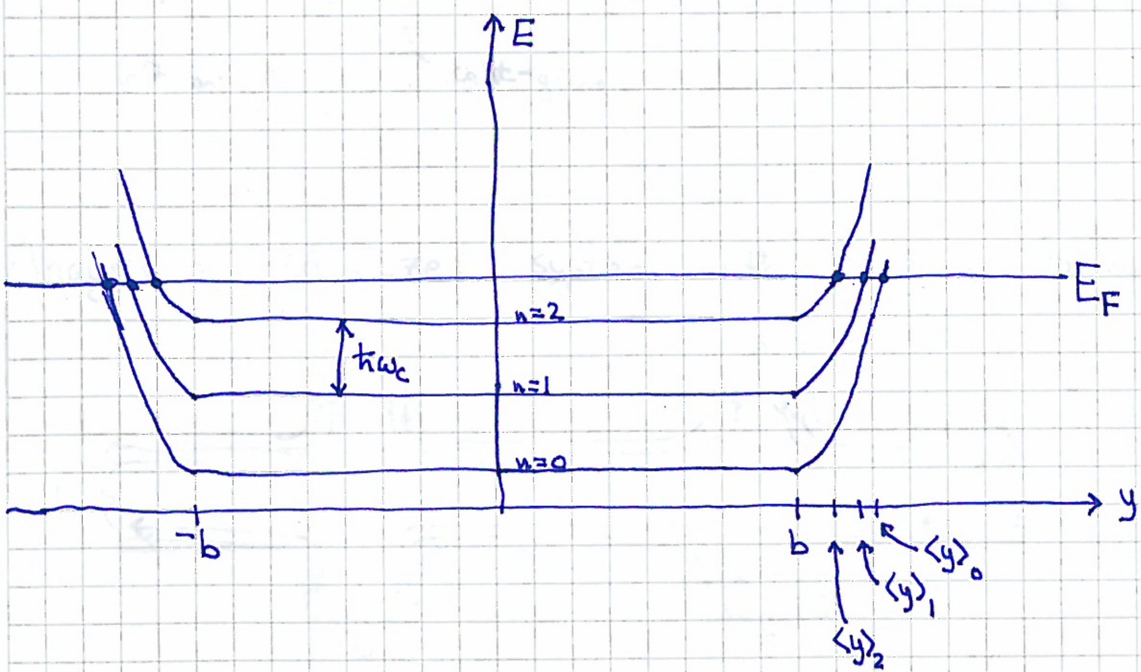
$$\left.\begin{aligned} E_n &= (n+\tfrac{1}{2})\hbar\omega_c \\ v_n &= \hbar^{-1}\,\partial E_n/\partial k = 0 \end{aligned}\right\} \text{ No transport. Electrons are circulating in cyclotron orbits.}$$

Edge region:

$$E_{nk} = (n+\tfrac{1}{2})\hbar\Omega + \frac{\hbar^2 k^2}{2M_B}$$

$$v_n(k) = \hbar k / M_B$$

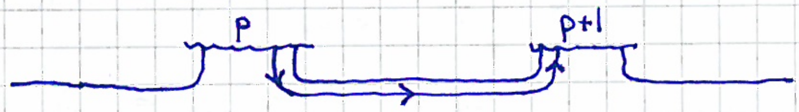
$y_{nk} = b + kL_B^2 = \langle y\rangle_n$ for edge state ψ_{nk} (at $E_{nk} = E_F$)



Most important consequence of this is the <u>Large separation</u> between right-going ($v_x > 0$) and left-going ($v_x < 0$) states

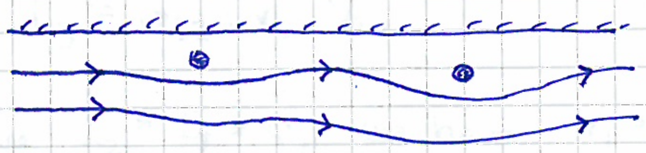
$\Rightarrow$ very small probability for back-scattering!

$\Rightarrow$ an electron that emerges from one terminal in an edge state has probability ≈ 1 for ending up in the neighbouring terminal:



This requires <u>ideal</u> terminals: $T_{p+1,p} \approx 1$

Robust with respect to impurity scattering:



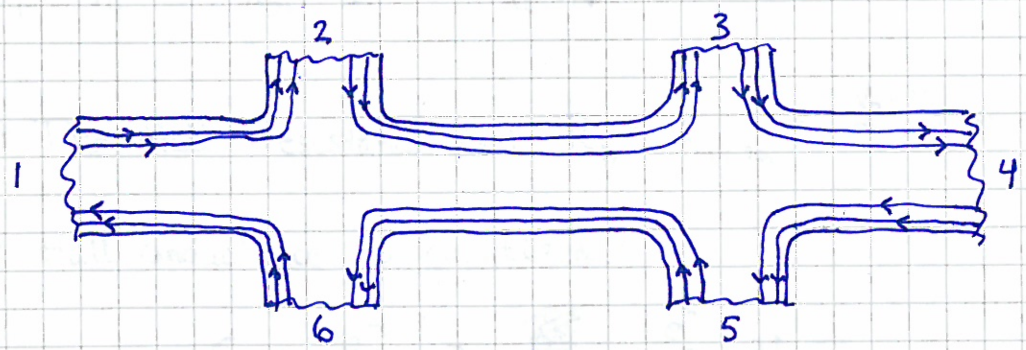
● = impurity; local potential $V_i(\vec{r})$

Back-scattering probability:

$$|\langle n | V_i | m \rangle|^2 \approx 0 \quad \text{(exponentially small)}$$

left-going   right-going

Analysis of idealized system with Büttiker formula:



2, 3, 5, 6 : Voltage probes $\Rightarrow I_2 = I_3 = I_5 = I_6 = 0$

1, 4 : Source and drain $\Rightarrow I = I_1 = I_{1\to4} = -I_4$

$V_1 - V_4 =$ applied voltage ("bias")

N edge states (in each direction) below E_F

$$\Rightarrow G = \frac{2e^2}{h} \cdot N$$

$$R_{14,14} = R_{2t} = \frac{1}{G} = \frac{h}{2e^2} \cdot \frac{1}{N} \quad \text{(2-term. resistance)}$$

Büttiker: $I_p = \sum_{q \neq p} G_{pq} (V_p - V_q)$

with $G_{21} = G_{32} = G_{43} = G_{54} = G_{65} = G_{16} = \frac{2e^2}{h} \cdot N$

$G_{12} = G_{23} = \ldots = 0$

$$\Rightarrow I_2 = G_{21} (V_2 - V_1)$$

$$I_3 = G_{32} (V_3 - V_2)$$

Since $I_2 = I_3 = 0$, we have $\underline{V_1 = V_2 = V_3}$

$$\Rightarrow \underline{R_L} = R_{14,23} = \frac{V_2 - V_3}{I} = \underline{\underline{0}}$$

[This is the classical result, which is correct for small $\vec{B}$.]

Similarly, since $I_5 = I_6 = 0$: $\underline{V_4 = V_5 = V_6}$

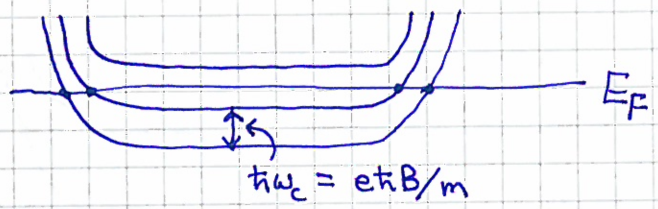
$$V_1 - V_4 = R_{2t} \cdot I = \frac{h}{2e^2 \cdot N} \cdot I$$

$\Rightarrow$ The Hall resistance is : $R_H = R_{14,26} = \frac{V_2 - V_6}{I} = \frac{V_1 - V_4}{I} = \frac{h}{2e^2 \cdot N}$

$\Rightarrow$ Hall conductance is quantized :

$$G_H = R_H^{-1} = \frac{2e^2}{h} \cdot N$$

How will G_H depend on B?



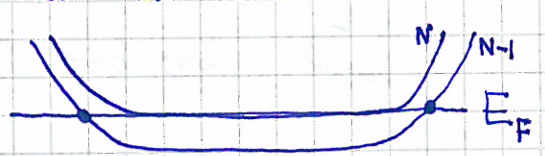
$$(N+1/2)\, e\hbar B/m \;<\; E_F \;<\; (N+3/2)\, e\hbar B/m$$

$$\Rightarrow N(B) \sim \left[\frac{m E_F}{e\hbar B}\right] = \left[\frac{\text{const}}{B}\right]$$

($[\ldots]$ = integer part of ....)

$\Rightarrow$ when B increases, $N \to N-1$ each time a Landau level rises above E_F

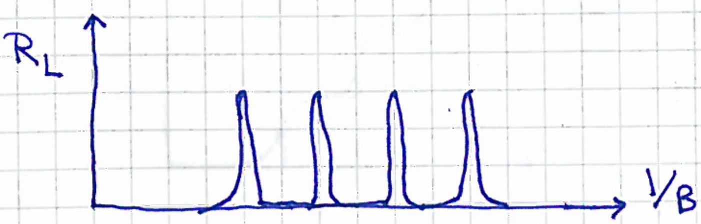
Near transition between two plateaus: $E_F = (N+1/2)\hbar\omega_c$



Edge States in LL $N-1$ are well separated, but LL N extends across sample in transverse direction

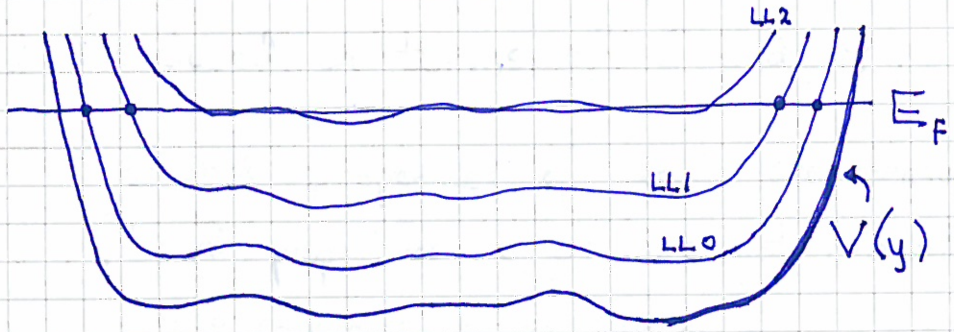
$\Rightarrow$ severely increased prob. for back-scattering in LL N

$\Rightarrow R_L \neq 0$ near transitions; $R_L \approx 0$ on "R_H-plateaus":



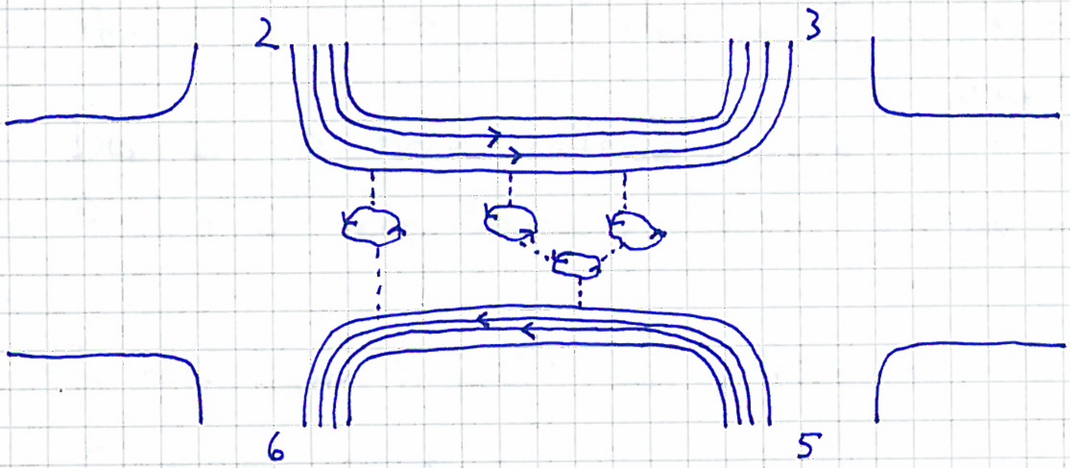
SdH oscillations, periodic in $1/B$

A more realistic situation:



$\Rightarrow$ both localized and extended states are possible at E_F

No back-scattering in LL0 and LL1.
Back-scattering possible in LL2 when $E^{bulk}_{LL2} \approx E_F$
Schematically:



$\Rightarrow$ reduced $G_{p+1,p}$ (e.g. G_{32}, G_{65})

$\Rightarrow V_2 > V_3$ etc.

$\Rightarrow R_L = (V_2 - V_3)/I > 0$

and $G_H < \frac{2e^2}{h} N$ when transition is approached: