NORGES TEKNISK-
NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR ENERGI- OG PROSESSTEKNIKK

Contact during exam:
Jon Andreas Støvneng
Phone: 73 59 36 63 / 45 45 55 33

# EXAM TEP4145 KLASSISK MEKANIKK

Monday 21. May 2007 kl. 0900 - 1300
English version

Remedies: C

- K. Rottmann: Mathematical formulae
- Approved calculator, with empty memory, according to list worked out by NTNU. (HP30S or similar.)

Appendix A: The questions (Page 2 - 6).
Appendix B: Formulas (Page 7).

The exam consists of 5 questions. The weight for each question is given. Do questions 1, 2 and 3, and either question 4 (physics) or 5 (cybernetics).
**Note** that Einstein's summation convention is used unless otherwise stated, i.e., summation over repeated indices.

The grades will be ready around June 12.

# Appendix A: The questions

**QUESTION 1** [Counts 25%]

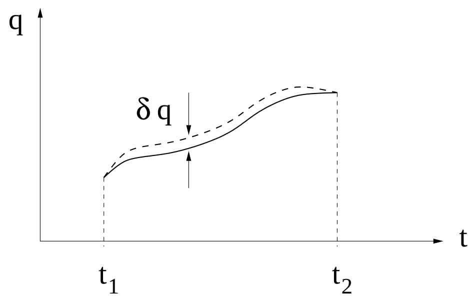
*a*) Derive Lagrange's equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

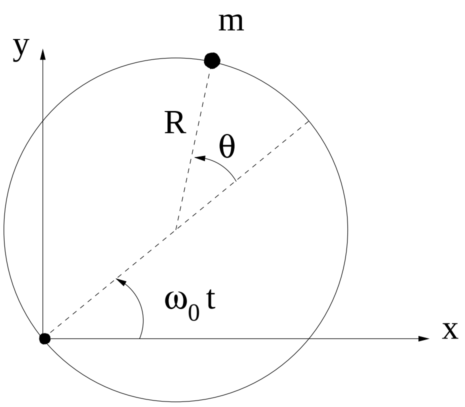
from Hamilton's principle

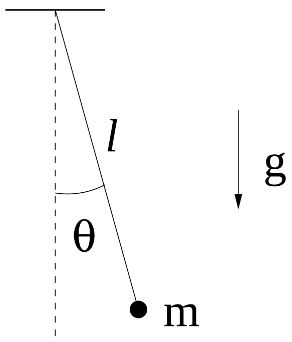
$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0.$$

for a system with one degree of freedom. Here, there is no variation in the end points, and virtual variations in the coordinate $q$ is performed at fixed time $t$, i.e., $\delta t = 0$.



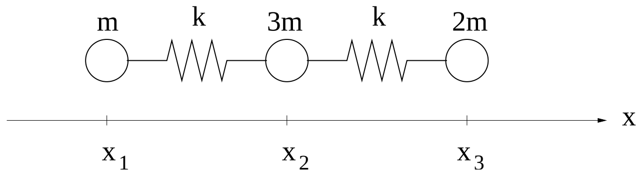

*b*) A point mass $m$ slides without friction on a ring which rotates counterclockwise in the $xy$ plane with constant angular velocity $\omega_0$. The ring, with radius $R$, rotates around a point on the ring. This point lies fixed at the origin $x = y = 0$, as shown in the figure below (to the left).
• Find the lagrangian $L(\theta, \dot{\theta})$ of the mass $m$.
• Find Lagrange's equation, i.e., the equation of motion for the mass $m$.
• The equation of motion would be the same if the mass $m$ were swinging back and forth at the end of a massless rod in the field of gravity (see figure below, to the right; acceleration of gravity = $g$). Verify this, and determine the length $l$ of the rod.



**QUESTION 2** [Counts 25%]

Three spheres of mass $m$ (sphere nr 1, to the left), $3m$ (sphere nr 2, in the middle), and $2m$ (sphere nr 3, to the right), respectively, are connected via two identical (and ideal) springs with spring constant $k$, as shown in the figure:




The springs can only be stretched, not bent. We assume that the spheres may only move along the $x$ axis, and we will here investigate oscillations around the equilibrium positions of the spheres. These are $x_{10}$, $x_{20}$, and $x_{30}$, respectively.

• Use the deviations from equilibrium, $\eta_i = x_i - x_{i0}$ ($i = 1, 2, 3$), as coordinates, and find the (symmetric) matrices $\boldsymbol{V}$ and $\boldsymbol{T}$ in the quadratic expressions

$$\begin{aligned} V &= \frac{1}{2} V_{ij}\,\eta_i\,\eta_j, \\ T &= \frac{1}{2} T_{ij}\,\dot{\eta}_i\,\dot{\eta}_j, \end{aligned}$$

for the potential and the kinetic energy of the system. Here, $V_{ij}$ and $T_{ij}$ are matrix elements of $\boldsymbol{V}$ and $\boldsymbol{T}$, respectively.

• Solve the secular equation

$$\left| \boldsymbol{V} - \omega^2\, \boldsymbol{T} \right| = 0,$$

and thus determine the two eigenfrequencies of the system, $f_\alpha = \omega_\alpha / 2\pi$ ($\alpha = 1, 2$). Find numerical values of $f_\alpha$ when $m = 100$ g and $k = 10^3$ N/m. (Neglect the mode with $\omega = 0$, which corresponds to pure translation of the system.)

**QUESTION 3** [Counts 25%.]

In this question, we will study a canonical transformation $(q,p) \to (Q,P)$ of a simple one-dimensional harmonic oscillator ($i = \sqrt{-1}$):

$$\begin{aligned} Q(q,p) &= \frac{1}{\sqrt{2i}}(q+ip)\,, \\ P(q,p) &= \frac{-1}{\sqrt{2i}}(q-ip)\,. \end{aligned}$$

For simplicity, we use $m = k = 1$ ($m$ = mass and $k$ = spring constant of the oscillator).

• What is the hamiltonian $H(q,p) = T + V$ of this oscillator?

• Show that the given transformation $(q,p) \to (Q,P)$ *is* canonical. Hint: Use the fact that Poisson brackets are invariant under a canonical transformation, for example $[q,p] = [Q,P]$.

• Find the hamiltonian $K(Q,P)$ in the new coordinates $Q, P$. (Here, $K = H$.)

• Find Hamilton's equations for $Q$ and $P$, and solve these equations, using the initial conditions $q(t=0) = p(t=0) = 1$.

• What is the total energy of the oscillator?

**QUESTION 4** [For the physics students, counts 25%.]

*a*) The electromagnetic field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}.$$

Find the elements of the matrix $\boldsymbol{F}$ (i.e.: expressed in terms of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$).

*b*) Show that $\boldsymbol{E} \cdot \boldsymbol{B}$ is invariant under a Lorentz transformation.

*c*) A point charge $q$ passes through the origin at time $t = 0$ and moves with constant velocity $v$ in the positive $x$ direction. Let us by $S_0$ denote the inertial system where the point charge is at rest, and by $S$ we denote the inertial system where the point charge moves with velocity $v\hat{x}$ ($\hat{x}$ is a unit vector). The electric field at a distance $\boldsymbol{r}$ from the point charge is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0} \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta)^{3/2}} \frac{\boldsymbol{r}}{r^3}$$

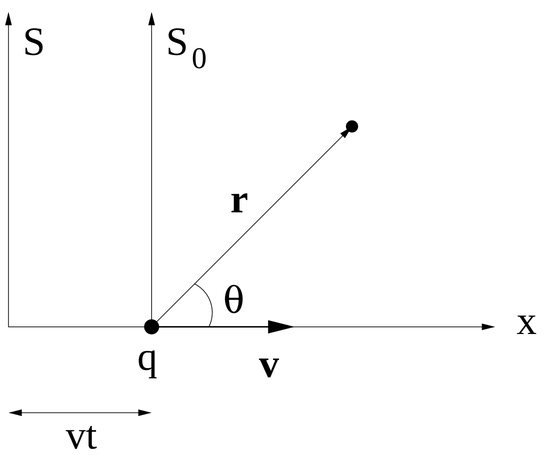
measured in $S$. Here, $\theta$ denotes the angle between the $x$ axis and $\boldsymbol{r}$, see figure below, and $\beta = v/c$. What is the magnetic field $\boldsymbol{B}_0$ measured in $S_0$? Show that the magnetic field, measured in $S$, may be written as

$$\boldsymbol{B} = \frac{1}{c^2}\, \boldsymbol{v} \times \boldsymbol{E}.$$

*d*) What is the magnetic field $\boldsymbol{B}$ in the nonrelativistic limit $v \ll c$? Compare the result with Biot-Savart's law,

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{I\, d\boldsymbol{l} \times \boldsymbol{r}}{r^3}.$$

Sketch field lines for $\boldsymbol{B}$ in the $yz$ plane.



**QUESTION 5** [For cybernetics students, counts 25%.]

*a*) Let us investigate how the stock of a certain animal will develop in time. Assume - in the simplest possible manner - that this number increases with a constant factor each year, i.e., there exists a $\mu > 1$ such that

$$b_{n+1} = \mu\, b_n,$$

where $b_n$ is the stock in year $n$ and $b_{n+1}$ is the stock in year $n+1$. What kind of equation is this, and what is its general solution? Explain briefly why this model is unrealistic.

*b*) An improved model is obtained by letting $B$ be an upper limit on the number of animals. Define the model by

$$b_{n+1} = \mu\, b_n \left(1 - \frac{b_n}{B}\right).$$

We introduce the relative stock,

$$x_n = \frac{b_n}{B},$$

and hence we have

$$x_{n+1} = \mu\, x_n \left(1 - x_n\right).$$

Next, we define

$$F_\mu(x) = \mu\, x \left(1 - x\right),$$

which yields

$$x_{n+1} = F_\mu(x_n).$$

Make a simple sketch of $F_\mu(x)$, and write down the coordinates of the maximum point of the curve. Let $x_0$ be the intersection between the graph $y = F_\mu(x)$ and the graph $y = x$. Then, what is $x_n$ as a function of $x_0$? What kind of point is $x_0$?

*c*) Define the concept "fixed point" for a function $f$. Define the concept "periodic point" for a function $f$, and explain what is meant by "fundamental period" for the point. Explain/define "attractive fixed point" and "repulsive fixed point" for a function $f$.

*d*) Finally, consider the function

$$F_\mu(x) = \mu\, x \left(1 - x\right).$$

Determine the two fixed points of $F_\mu$. Let the upper limit of $\mu$ be 3, and determine whether the fixed points are attractive or repulsive.

# Appendix B: Formulas

The validity of the formulas and the meaning of the symbols are assumed to be known.

- Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

- Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- Poisson brackets:

$$[f,g] = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$$

$$[q_i, q_j] = [p_i, p_j] = 0 \quad , \quad [q_i, p_j] = \delta_{ij}$$

- Four-vector:

$$x_\mu = (\boldsymbol{r}, ict)$$

- Four-potential:

$$A_\mu = (\boldsymbol{A}, i\phi/c)$$

- Electromagnetic field:

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

- Lorentz transformation (with relative velocity $\boldsymbol{v} = v\hat{x}$):

$$L_{22} = L_{33} = 1 \;\; , \;\; L_{11} = L_{44} = \gamma \;\; , \;\; L_{14} = -L_{41} = i\beta\gamma$$

$$\beta = v/c \quad , \quad \gamma = 1/\sqrt{1 - v^2/c^2}$$

- Lorentz transformation of electromagnetic field (the inertial system $S_0$ moves with velocity $v\hat{x}$ relative to $S$):

$$\begin{aligned} E_x &= E_{0x} \quad , \quad E_y = \gamma(E_{0y} + vB_{0z}) \quad , \quad E_z = \gamma(E_{0z} - vB_{0y}) \\ B_x &= B_{0x} \quad , \quad B_y = \gamma(B_{0y} - \frac{v}{c^2}E_{0z}) \quad , \quad B_z = \gamma(B_{0z} + \frac{v}{c^2}E_{0y}) \end{aligned}$$

- Trigonometric relations:

$$\begin{aligned} \cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\ \sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \end{aligned}$$