Massachusetts Institute of Technology

Physics 8.03

Exam 1

Thursday, October 14, 2004

- You have 85 minutes
- There are FOUR problems
- You may use calculators
- This is a closed-book exam; no notes are allowed.

Useful Formulae

General differential equation for oscillators:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f\,\cos(\omega t)$$

has solutions

$$\begin{aligned}
x(t) &= A\,\mathrm{e}^{-\frac{\gamma t}{2}}\,\cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\; t + \alpha\right) + x_{ss}(t) \qquad && \omega_0 > \frac{\gamma}{2} \\
x(t) &= (A + B\,t)\,\mathrm{e}^{-\frac{\gamma t}{2}} + x_{ss}(t) && \omega_0 = \frac{\gamma}{2} \\
x(t) &= A\,\mathrm{e}^{-\Gamma_1 t} + B\,\mathrm{e}^{-\Gamma_2 t} + x_{ss}(t) && \omega_0 < \frac{\gamma}{2}
\end{aligned}$$

where

$$\Gamma_{\frac{1}{2}} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

and

$$x_{ss}(t) = A(\omega)\,\cos(\omega t - \delta(\omega))$$

$$A(\omega) = \frac{f}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2\right]^{1/2}} \qquad\qquad \tan\delta(\omega) = \frac{\gamma\,\omega}{\omega_0^2 - \omega^2}$$

Complex exponentials:

$$\mathrm{e}^{j\,\theta} = \cos\theta + j\,\sin\theta \qquad \sin\theta = \frac{\mathrm{e}^{j\,\theta} - \mathrm{e}^{-j\,\theta}}{2\,j} \qquad \cos\theta = \frac{\mathrm{e}^{j\,\theta} + \mathrm{e}^{-j\,\theta}}{2}$$

Trigonometric Formulae:

$$\begin{aligned}
\sin(a+b) &= \sin a\,\cos b + \cos a\,\sin b \\
\cos(a+b) &= \cos a\,\cos b - \sin a\,\sin b \\
\sin a + \sin b &= 2\,\sin\left(\frac{a+b}{2}\right)\,\cos\left(\frac{a-b}{2}\right) \\
\sin a - \sin b &= 2\,\cos\left(\frac{a+b}{2}\right)\,\sin\left(\frac{a-b}{2}\right) \\
\cos a + \cos b &= 2\,\cos\left(\frac{a+b}{2}\right)\,\cos\left(\frac{a-b}{2}\right) \\
\cos a - \cos b &= -2\,\sin\left(\frac{a+b}{2}\right)\,\sin\left(\frac{a-b}{2}\right)
\end{aligned}$$

Non-dispersive wave equation:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2}$$

where $v = \sqrt{\frac{T}{\mu}}$ for a string and $v = \sqrt{\frac{\kappa}{\rho}} = \sqrt{\frac{R\,T\,\gamma}{M}}$ for a gas.

Kinetic, potential energy and power:

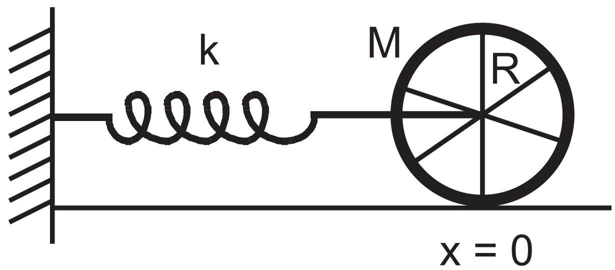
$$\frac{dK}{dx} = \frac{1}{2}\,\mu\left(\frac{\partial y}{\partial t}\right)^2 \qquad \frac{dU}{dx} = \frac{1}{2}\,T\left(\frac{\partial y}{\partial x}\right)^2 \qquad P(t) = -T\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial y}{\partial x}\right)$$

Reflection and transmission coefficients:

$$R = \frac{v_2 - v_1}{v_2 + v_1} \qquad T = \frac{2v_2}{v_2 + v_1}$$

Problem 1 (25 pts): Simple harmonic oscillator

The hub of a wheel is attached to a spring with spring constant k and negligible mass. The wheel has radius R and total mass M. The mass of the spokes is negligibly small. The wheel rolls *without slipping*, i.e., the wheel translates by the same distance that its circumference rotates. The center of mass of the wheel oscillates (simple harmonic motion) in the horizontal direction about its equilibrium point $x = 0$.



a. (10 pts) Find an expression for the total energy in terms of k, M, R and $x(t)$. Since the spokes have negligible mass, you may assume that the moment of inertia for rotation about the axle is MR^2.

b. (10 pts) Using conservation of energy, derive the differential equation of motion.

c. (5 pts) What is the angular frequency of small oscillations about equilibrium?

Problem 2 (20 pts): Boundary conditions in a pipe

Pressure oscillations in a hollow pipe (filled with air) of length L are described by the wave equation

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}$$

where positive p is the over-pressure (over and above the one atmosphere ambient pressure), and z is the longitudinal direction along the pipe. The pipe is closed at one end and open at the other.

a. (5 pts) What is the meaning of v, and what is its approximate value (in m/sec)?

b. (10 pts) Assuming a solution of the form

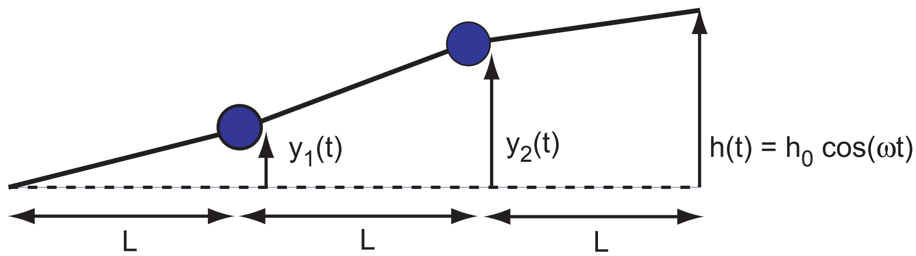
$$p(z,t) = [A\ coskz + B\ sinkz]cos\omega t$$

find all the unknowns ($A,\ B,\ k$ and ω) if $p(z=0, t=0) = p_0$. The closed end of the pipe is at $z = 0$, and the open end at $z = L$.

c. (5 pts) For $L = 0.5\,m$, what are the approximate frequencies (in Hz), and what are the approximate wavelengths (in m) of the first and second harmonic? (The first harmonic is also called the fundamental).

Problem 3 (30 pts): Driven coupled oscillators

Two identical beads, each of mass M are placed at equal spacing along a string of length $3\,L$ and negligible mass, under a uniform tension τ. Any form of damping can be ignored. One end of the string is firmly attached to a rigid wall, but the other end is harmonically driven with a small transverse displacement, $h(t) = h_0 \cos(\omega\, t)$.



a. (10 pts) Taking the transverse displacement from equilibrium of each bead to be $y_1(t)$ and $y_2(t)$, respectively, derive the equations of motion for each mass.

b. (10 pts) Find the angular frequencies of the normal modes of oscillation.

c. (5 pts) Derive the steady-state amplitude of each mass as a function of driving frequency, ω. What are their values for $\omega = 0$?

d. (5 pts) Make a sketch of these amplitudes as a function of ω. Plot an amplitude as positive when it is in phase with the driver, and negative when it is out of phase with the driver. Either make two different plots, or indicate very clearly which curves go with bead #1 and which with bead #2. Your plots should reflect the values of the amplitudes for $\omega = 0$.

Problem 4 (25 pts): Phase jump

A weakly damped harmonic oscillator (characterized by a damping constant $\gamma \ll \omega_0$) of mass m is driven by an externally applied force $F(t)$ (you may assume a mass on a spring with the external force $F(t)$ acting on the mass). The driving frequency ω is identical to the resonance frequency ω_0 of the oscillator.

a. (5 pts) What is the displacement $x(t)$ of the driven oscillator in steady state for $F(t) = F_0 \cos(\omega_0 t)$?

b. (10 pts) If the external force is $F(t) = F_0 \sin(\omega_0 t)$, what then is $x(t)$ in steady state?

c. (10 pts) Now assume that the external force is

$$F(t) = \begin{cases} F_0 \cos(\omega_0 t) & (t \leq 0) \\ F_0 \sin(\omega_0 t) & (t > 0) \end{cases}$$

What is $x(t)$ for $t > 0$?
You should assume that for $t \leq 0$ the system oscillates with its steady state solution for the external force $F(t) = F_0 \cos(\omega_0 t)$.
Hint: The values of x and the velocity at time $t = 0$ follow from the steady state solution in part (a). For $t > 0$ there will be a new steady state, plus a transient.