Massachusetts Institute of Technology

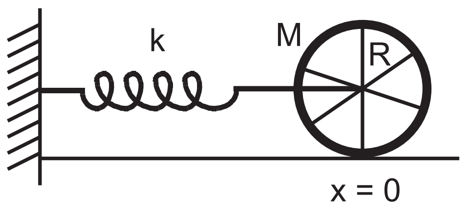
Physics 8.03

Exam 1 Solutions

Thursday, October 14, 2004

Problem 1

Without slipping: when the center of the wheel moves a distance x, the wheel has rotated an angle $\theta = \frac{x}{2\pi R}2\pi = \frac{x}{R}$ radians. Angular velocity $\omega = \dot{\theta} = \frac{\dot{x}}{R}$, thus the velocity of the center of the wheel $v = \omega R$.



(a) The total energy at x point is

$$E_{tot}(x) = \frac{1}{2}kx^2 + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}kx^2 + \frac{1}{2}M\dot{x}^2 + \frac{1}{2}MR^2\frac{\dot{x}^2}{R^2} = \frac{1}{2}kx^2 + M\dot{x}^2 \quad (1)$$

(b) Time derivative $dE/dt = 0$ gives

$$2M\ddot{x} + kx = 0 \quad (2)$$

(c) The angular frequency is

$$\omega = \sqrt{\frac{k}{2M}} \quad (3)$$

Problem 2

(a) $v \approx 340$ m/sec is the speed of sound in air (measured during lecture on Oct. 5).

(b) The sound wave in this pipe is shown in the following figure. Where ξ represents the position of the air molecules, and $p \propto \frac{\partial \xi}{\partial z}$ is the pressure over and above 1 atm.

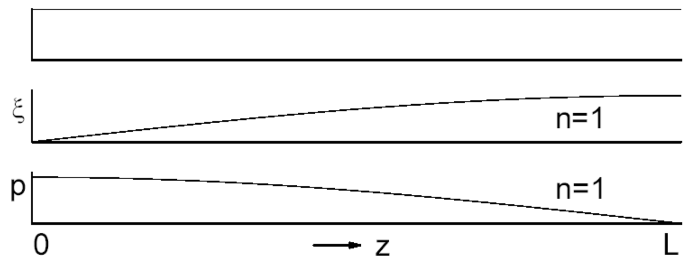
At the closed end there is always a node in ξ and an anti-node in p. At the open end, there is always an anti-node in ξ and a node in p. Thus, in p space we only have the cos term because $B = 0$.

The boundary conditions are

$$p(z = 0, t = 0) = p_0 = A \quad (4)$$

and at any time t,

$$p(z = L, t) = 0 = p_0 \cos k_n L \quad (5)$$



Therefore

$$k_n = \frac{(2n-1)\pi}{2L} \tag{6}$$

NOTICE: at any time, $p(z = L, t) = 0$ thus

$$0 = p_0 \cos k_n L + B \sin k_n L \Rightarrow B = \frac{-p_0}{\tan(k_n L)}$$

$\tan(k_n L) = \pm\infty$ for any n. This is consistent with $B = 0$.

And

$$\omega = v k_n = \frac{(2n-1)\pi v}{2L} \tag{7}$$

This can be found by substituting $p(z,t) = p_0 \cos kz \cos \omega t$ into the wave equation.

$$\begin{aligned}
\frac{\partial p}{\partial z} &= -p_0 k \sin kz \cos \omega t && (8)\\
\frac{\partial^2 p}{\partial z^2} &= -p_0 k^2 \cos kz \cos \omega t && (9)\\
\frac{\partial p}{\partial t} &= -p_0 \omega \cos kz \sin \omega t && (10)\\
\frac{\partial^2 p}{\partial t^2} &= -p_0 \omega^2 \cos kz \cos \omega t && (11)
\end{aligned}$$

Thus

$$-p_0 k^2 \cos kz \cos \omega t = -\frac{1}{v^2} p_0 \omega^2 \cos kz \cos \omega t \tag{12}$$

which leads to

$$k^2 = \frac{\omega^2}{v^2} \Rightarrow \omega = vk \tag{13}$$

(c) $L = 0.5$ cm, $k_1 = \pi \Rightarrow \lambda_1 = \frac{2\pi}{k_1} = 2$ m $(\lambda_1 = 4L)$.
$\lambda_1 = \frac{v}{f_1} \Rightarrow f_1 = \frac{2\pi}{\omega_1} = \frac{340}{2} = 170$ Hz.
$k_2 = 3\pi \Rightarrow \lambda_2 = \frac{2\pi}{k_2} = \frac{2}{3}$ m $\Rightarrow f_2 = \frac{340}{2/3} = 510$ Hz $= 3f_1$.

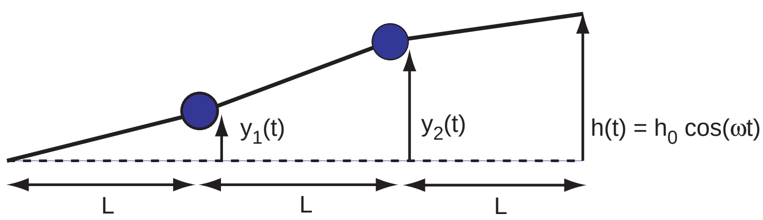
Problem 3

(a) The equations of motion for each mass are

$$M\,\ddot{y}_1 = -\tau\,\sin\theta_1 + \tau\,\sin\theta_2 \tag{14}$$
$$M\,\ddot{y}_2 = -\tau\,\sin\theta_2 + \tau\,\sin\theta_3 \tag{15}$$

Using the small angle approximation that $\sin\theta \approx \tan\theta$, we get

$$\sin\theta_1 \approx \frac{y_1}{L} \qquad \sin\theta_2 \approx \frac{y_2 - y_1}{L} \qquad \sin\theta_3 \approx \frac{y_2 - h}{L}$$



Taking $h(t) = h_0\,\cos\omega\,t$ and $\omega_0^2 = \dfrac{\tau}{M\,L}$, the equations of motion are

$$\ddot{y}_1 + 2\,\omega_0^2\,y_1 - \omega_0^2\,y_2 = 0 \tag{16}$$
$$\ddot{y}_2 + 2\,\omega_0^2\,y_2 - \omega_0^2\,y_1 = \omega_0^2\,h_0\,\cos\omega\,t \tag{17}$$

(b) Inserting trial solutions of the form $y_i(t) = C_i\,\cos\omega\,t$ into the differential equations, we get

$$\begin{pmatrix} (2\,\omega_0^2 - \omega^2) & -\omega_0^2 \\ -\omega_0^2 & (2\,\omega_0^2 - \omega^2) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_0^2\,h_0 \end{pmatrix} \tag{18}$$

The frequencies of the normal modes are given by the roots of the determinant of the matrix, i.e.

$$\left(2\,\omega_0^2 - \omega^2\right)^2 - \omega_0^4 = 0 \quad \Rightarrow \quad 2\,\omega_0^2 - \omega^2 = \pm\omega_0^2 \tag{19}$$
$$\Rightarrow \quad \omega_L^2 = \omega_0^2 \tag{20}$$
$$\omega_H^2 = 3\,\omega_0^2 \tag{21}$$

(c) Using Cramer's rule,

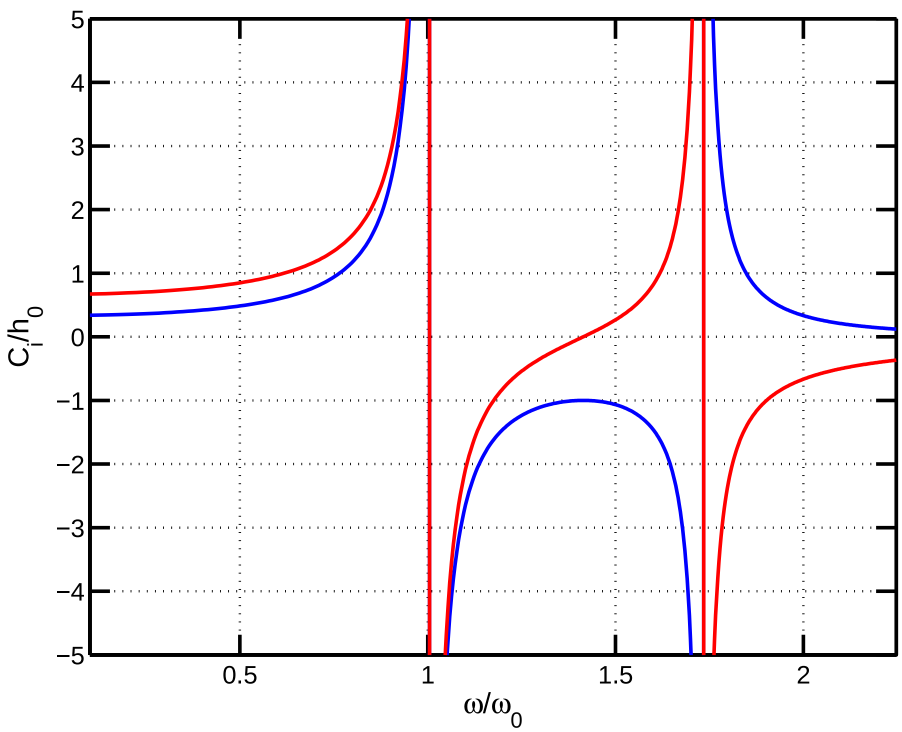
$$C_1 = \frac{\begin{vmatrix} 0 & -\omega_0^2 \\ -\omega_0^2\,h_0 & (2\,\omega_0^2 - \omega^2) \end{vmatrix}}{\left(\omega_0^2 - \omega^2\right)\,\left(3\,\omega_0^2 - \omega^2\right)} = \frac{\omega_0^4\,h_0}{\left(\omega_0^2 - \omega^2\right)\,\left(3\,\omega_0^2 - \omega^2\right)} \tag{22}$$

$$C_2 = \frac{\begin{vmatrix} (2\,\omega_0^2 - \omega^2) & 0 \\ -\omega_0^2 & -\omega_0^2\,h_0 \end{vmatrix}}{\left(\omega_0^2 - \omega^2\right)\,\left(3\,\omega_0^2 - \omega^2\right)} = \frac{\omega_0^2\,h_0\,\left(2\,\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)\,\left(3\,\omega_0^2 - \omega^2\right)} \tag{23}$$

At $\omega = 0$, $C_1 = \dfrac{1}{3}\,h_0$ and $C_2 = \dfrac{2}{3}\,h_0$.

NOTICE: at $\omega = \sqrt{2}\omega_o, C_1 = -h_o$, and $C_2 = 0$. Object 2 stands still while object 1 is oscillating with amplitude h_o 180 degrees out of phase with the driver. Rather strange, isn't it? This bizarre behavior was demonstrated in three different ways in lectures.

(d) C_1 is plotted in blue and C_2 is the red curve in the figure.



Problem 4

Steady state solution:

(a) $x(t) = x_0 \cos(\omega t - \delta)$

$$x_0 = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2]^{1/2}} \qquad tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \tag{24}$$

Here $\omega = \omega_0 \Rightarrow x_0 = \frac{F_0}{m\omega_0\gamma}$ and $\delta = \frac{\pi}{2}$.

$$x(t) = \frac{F_0}{m\omega_0\gamma}\cos(\omega_0 t - \frac{\pi}{2}) = \frac{F_0}{m\omega_0\gamma}\sin\omega_0 t = x_0 \sin\omega_0 t \tag{25}$$

(b) $x(t) = x_0 \sin(\omega_0 t - \delta)$, all else is the same. Thus

$$x(t) = \frac{F_0}{m\omega_0\gamma}\sin(\omega_0 t - \frac{\pi}{2}) = \frac{-F_0}{m\omega_0\gamma}\cos\omega_0 t = -x_0\cos\omega_0 t \tag{26}$$

(c) $t > 0$ General solution: transient + steady state $\omega = \omega_0$.

$$x(t) = \frac{-F_0}{m\omega_0\gamma}\cos\omega_0 t + Ae^{-\frac{\gamma}{2}t}\cos(\omega' t + \alpha) \qquad \omega'^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{27}$$

The first term is the steady state solution and the second term is the transient part.
Initial condition: $x(t=0)$ follows from the steady state solution Eq. (25) prior to $t = 0$. That steady state solution: $x(t) = \frac{F_0}{m\omega_0\gamma}\sin\omega_0 t = x_0\sin\omega_0 t \Rightarrow x(t=0) = 0$. Important!
From the general solution expression Eq. (27)

$$x(t=0) = 0 \Rightarrow -x_0 + A\cos\alpha = 0 \Rightarrow x_0 = A\cos\alpha \tag{28}$$

$\dot{x}(t=0)$ also follows from the steady state solution Eq. (25) prior to $t = 0$.

$$\dot{x} = x_0\omega_0\cos\omega_0 t \Rightarrow \dot{x}(t=0) = x_0\omega_0 \tag{29}$$

The time derivative of the general solution Eq. (27)

$$\dot{x} = \omega_0 x_0\sin\omega_0 t - \frac{\gamma}{2}Ae^{-\gamma t/2}\cos(\omega' t + \alpha) - Ae^{-\gamma t/2}\omega'\sin(\omega' t + \alpha)$$

Matching the initial condition $\dot{x}(t=0) = x_0\omega_0$ gives us

$$\dot{x}(x=0) = x_0\omega_0 = \frac{-\gamma}{2}A\cos\alpha - A\omega'\sin\alpha \tag{30}$$

Noticing from Eq. (28) we have $A = \frac{x_0}{\cos\alpha}$, therefore

$$\omega_0 = -\frac{\gamma}{2} - \omega'\tan\alpha \tag{31}$$

and

$$\tan\alpha = \frac{-(\omega_0 + \frac{\gamma}{2})}{\omega'} \tag{32}$$

Since $\gamma \ll \omega_0, \omega' \approx \omega_0$, thus $\tan\alpha \approx -1, \alpha \approx 45°$.

$$A = \frac{F_0}{m\omega_0\gamma\cos\alpha} \approx \frac{\sqrt{2}F_0}{m\omega_0\gamma} \tag{33}$$

Alternatively, instead of Eq. (27) we could have said

$$x(t) = -x_0\cos\omega_0 t + Ae^{-\frac{\gamma}{2}t}\sin\omega' t + Be^{-\frac{\gamma}{2}t}\cos\omega' t \tag{34}$$

$x(t=0) = 0 \Rightarrow -x_0 + B = 0 \Rightarrow B = x_0$.

$$\begin{aligned}\dot{x}(t) &= \omega_0 x_0\sin\omega_0 t - A\frac{\gamma}{2}e^{\frac{-\gamma}{2}t}\sin\omega' t + A\omega' e^{\frac{-\gamma}{2}t}\cos\omega' t \\ &\quad - B\frac{\gamma}{2}e^{\frac{-\gamma}{2}t}\cos\omega' t - B\omega' e^{\frac{-\gamma}{2}t}\sin\omega' t\end{aligned} \tag{35}$$

$\dot{x}(t=0) = x_0\omega_0 = A\omega' - x_0\frac{\gamma}{2} \Rightarrow A = x_0(\omega_0 + \frac{\gamma}{2})/\omega'$.
Since $\gamma \ll \omega_0, \omega' \approx \omega_0$, thus $A \approx x_0, B = \frac{F_0}{m\omega_0\gamma} \approx A$.