Useful Formulae

General differential equation for oscillators

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

has solutions

$$\begin{aligned}
x(t) &= A\,e^{-\frac{\gamma t}{2}} \cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\; t + \alpha\right) &+ x_{ss}(t) \qquad & \omega_0 > \frac{\gamma}{2} \\
x(t) &= (A + B\,t)\; e^{-\frac{\gamma t}{2}} &+ x_{ss}(t) \qquad & \omega_0 = \frac{\gamma}{2} \\
x(t) &= A\,e^{-\Gamma_1 t} + B\,e^{-\Gamma_2 t} &+ x_{ss}(t) \qquad & \omega_0 < \frac{\gamma}{2}
\end{aligned}$$

where

$$\Gamma_{\frac{1}{2}} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

and the steady-state solution is

$$x_{ss}(t) = A(\omega)\, \cos(\omega t - \delta(\omega))$$

$$A(\omega) = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2\right]^{1/2}} \qquad\qquad \tan\delta(\omega) = \frac{\gamma\,\omega}{\omega_0^2 - \omega^2}$$

Non-dispersive wave equation

$$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2} y(x,t)$$

where $v = \sqrt{T/\mu}$ for a string or $v = \sqrt{\kappa/\rho}$ for a gas.

Kinetic, potential energy and power

$$\frac{dK}{dx} = \frac{1}{2}\,\mu\left(\frac{\partial y}{\partial t}\right)^2 \qquad \frac{dU}{dx} = \frac{1}{2}\,T\left(\frac{\partial y}{\partial x}\right)^2 \qquad P(t) = -T\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial y}{\partial x}\right)$$

Reflection and transmission coefficients

$$R = \frac{v_2 - v_1}{v_2 + v_1} \qquad T = \frac{2v_2}{v_2 + v_1}$$

Fourier series for a function $f(\theta) = f(\theta + 2\pi)$

$$
\begin{aligned}
f(\theta) &= \sum_{m=1}^{\infty} \left[\frac{A_0}{2} + A_m \cos(m\,\theta) + B_m \sin(m\,\theta) \right] \\
A_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(m\theta)\, d\theta \qquad m = 0, 1, 2, \ldots \\
B_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(m\theta)\, d\theta \qquad m = 1, 2, 3, \ldots
\end{aligned}
$$

Dispersion

$$
v_{phase} = \frac{\omega}{k} \qquad \text{and} \qquad v_{group} = \frac{\mathrm{d}\omega}{\mathrm{d}k}
$$

Maxwell's equations

$$
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
$$

$$
\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0\, \epsilon_0\, \frac{\partial \vec{E}}{\partial t} + \mu_0\, \vec{J}
$$

EM force, flux, energy, intensity

$$
\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \qquad \vec{S} = \tfrac{1}{\mu_0}\, \vec{E} \times \vec{B}
$$

$$
U_E = \frac{\epsilon_0}{2} \left| \vec{E} \right|^2 \qquad U_M = \frac{1}{2\,\mu_0} \left| \vec{B} \right|^2
$$

Dipole approximation

$$
\begin{aligned}
\vec{E}_{rad}(\vec{r}, t) &= \frac{-q\vec{a}_{\perp}(t - r/c)}{4\pi\epsilon_0 c^2 r} && \text{Volt/m} \\
\vec{B}_{rad}(\vec{r}, t) &= \frac{1}{c}\, \hat{r} \times \vec{E}_{rad}(t) && \text{Tesla} \\
\vec{S}_{rad}(\vec{r}, t) &= \frac{1}{\mu_0} \vec{E}_{rad} \times \vec{B}_{rad} && \text{Watt/m}^2 \\
P &= \frac{q^2 a^2}{6\pi\epsilon_0 c^3} && \text{Watt}
\end{aligned}
$$

Boundary conditions at the surface of a perfect conductor

$$
E_{//} = 0 \qquad \left| B_{//} \right| = \mu_0\, |J_S|
$$

$$
E_{\perp} = \frac{\rho_S}{\epsilon_0} \qquad B_{\perp} = 0
$$

Transmission lines

$$\frac{\partial V}{\partial z} = -L_0 \frac{\partial I}{\partial t} \qquad v_p = \frac{1}{\sqrt{L_0 C_0}} \qquad \frac{V_r}{V_i} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\frac{\partial I}{\partial z} = -C_0 \frac{\partial V}{\partial t} \qquad Z_0 = \sqrt{\frac{L_0}{C_0}} \qquad \frac{I_r}{I_i} = \frac{Z_0 - Z_L}{Z_L + Z_0}$$

Boundary conditions at the surface of a perfect dielectric

$$E_{//}^{(1)} = E_{//}^{(2)} \qquad \frac{B_{//}^{(1)}}{\mu_1} = \frac{B_{//}^{(2)}}{\mu_2}$$

$$\kappa_{e1} E_{\perp}^{(1)} - \kappa_{e2} E_{\perp}^{(2)} = \frac{\rho_s}{\epsilon_0} \qquad B_{\perp}^{(1)} = B_{\perp}^{(2)}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Fresnel equations

$$r_{\parallel} = E_{0r\parallel}/E_{0i\parallel} = \frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} = -\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}$$

$$r_{\perp} = E_{0r\perp}/E_{0i\perp} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}$$

$$t_{\parallel} = E_{0t\parallel}/E_{0i\parallel} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} = \frac{2 \sin\theta_2 \cos\theta_1}{\sin(\theta_1 + \theta_2) \cos(\theta_1 - \theta_2)}$$

$$t_{\perp} = E_{0t\perp}/E_{0i\perp} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} = \frac{2 \sin\theta_2 \cos\theta_1}{\sin(\theta_1 + \theta_2)}$$

Special case of normal incidence ($\theta_1 = \theta_2 = 0$)

$$r_{\parallel,\perp} = \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} \qquad t_{\parallel,\perp} = \frac{E_t}{E_i} = \frac{2\,n_1}{n_1 + n_2}$$

Doppler Effect

$$\frac{\lambda'}{\lambda} = \frac{1 - \beta \cos\theta}{\sqrt{1 - \beta^2}} \qquad \text{for EM waves}$$

$$\frac{f'}{f} = \frac{v_s + v_r \cos\theta_r}{v_s - v_t \cos\theta_t} \qquad \text{for sound waves}$$

N source interference and diffraction:

Interference $$I = I_0 \left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2 \qquad \delta = \frac{2\pi}{\lambda} d \sin\theta$$

Diffraction $$I = I_0 \left[\frac{\sin\beta}{\beta}\right]^2 \qquad \beta = \frac{\pi}{\lambda} D \sin\theta$$

Diffraction gratings:

$$I = I_0 \left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2 \left[\frac{\sin\beta}{\beta}\right]^2$$

Physical constants

Speed of light	c	3×10^8	$\mathrm{m\ s^{-1}}$
Vacuum permeability	μ_0	1.26×10^{-6}	$\left(\mathrm{V\ m^{-1}}\right)/\mathrm{A}$
Vacuum permittivity	ϵ_0	8.85×10^{-12}	$\mathrm{C}/\left(\mathrm{V\,m^{-1}}\right)$
Electron rest mass	m	9.1×10^{-31}	kg
Elementary charge	e	1.6×10^{-19}	C
Gravitational constant	G	6.7×10^{-11}	$\mathrm{N\,m^2/kg^2}$

Trigonometric Formulae

$$\begin{aligned}
\sin(a+b) &= \sin a\cos b + \cos a \sin b \\
\cos(a+b) &= \cos a\cos b - \sin a \sin b \\
\sin a + \sin b &= 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \\
\sin a - \sin b &= 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) \\
\cos a + \cos b &= 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \\
\cos a - \cos b &= -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\sin(\theta \pm \frac{\pi}{2}) &= \pm\cos\theta \\
\cos(\theta \pm \frac{\pi}{2}) &= \mp\sin\theta \\
\sin(\theta \pm \pi) &= -\sin\theta \\
\cos(\theta \pm \pi) &= -\cos\theta
\end{aligned}$$

Complex exponentials

$$\mathrm{e}^{j\,\theta} = \cos\theta + j\,\sin\theta \qquad \sin\theta = \frac{\mathrm{e}^{j\,\theta} - \mathrm{e}^{-j\,\theta}}{2\,j} \qquad \cos\theta = \frac{\mathrm{e}^{j\,\theta} + \mathrm{e}^{-j\,\theta}}{2}$$