Intuitiv forklaring på at gruppe-hastigheten til en bølge, v_g, er gitt ved $\frac{d\omega(k)}{dk}$.

Bølge sammensatt av Fourier komponenter:

$$D(x,t) = \int dk \left\{ a(k)\cos(kx-\omega(k)t) + b(k)\sin(kx-\omega(k)t) \right\}$$

Fourier komponenter.

Ønsker å finne hastigheten til områdene i bølgen hvor amplituden er stor.

Må finne områdene langs x-aksen hvor vi har konstruktiv interferens.

M.a.o. vi ønsker å finne $x = x(t)$ slik at fasene $(k\,x(t) - \omega(k)\,t)$ ikke endrer seg mye når vi summerer over k (og Fourier-koeffisientene $a(k)$ og $b(k)$ ikke er funksjoner som endrer seg raskt over k).

Fasen endrer seg minimalt når 1. deriverte av den med hensyn på k er null:

$$\frac{d}{dk}\left\{ k\,x(t) - \omega(k)\,t \right\} = 0$$

eller

$$x(t) = \frac{d\omega(k)}{dk}\,t\,.$$

Deriverer m.h.p. tiden:

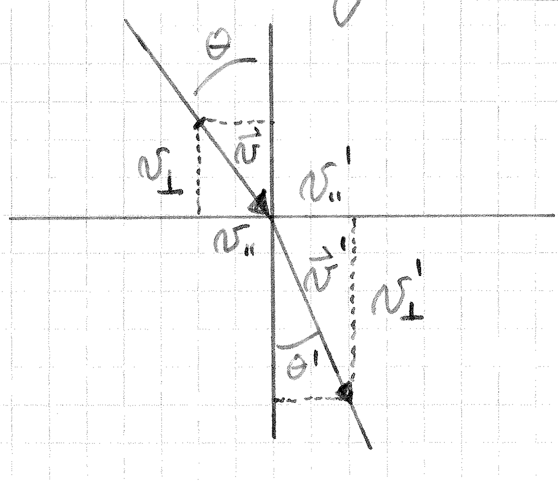
$$\boxed{\frac{dx(t)}{dt} = \frac{d\omega(k)}{dk}}$$

Dette er gruppehastigheten og den forteller oss altså hvordan områdene med konstruktiv interferens beveger seg.

Isaac Newton forklarte i "Opticks" (1704) Snells brytningslov ved hjelp av sin "partikkelmodell" for lysstråler.

Vi har tidligere utledet Snells brytningslov generelt for bølger.

Her er Newtons forklaring:



For en gitt lysstråle (det vil si med gitt f og hastighet $\vec{v}$) som kommer inn mot en grenseflate mot et annet medium og brytes inn i dette antaes:

Lyspartikkelen blir påvirket av en kraft ⊥ på grenseplanet når den passerer grensen. Det er ingen kraft parallell med grenseflaten.

For brytning mot innfallsloddet er denne kraften slik at $v_\perp' > v_\perp$.

For parallellkomponentene av hastigheten har vi $v_\parallel' = v_\parallel$ siden det er ingen kraft parallelt med grenseplanet.

Derimed har vi

$$v \sin\theta = v' \sin\theta'$$

som gir

$$\frac{\sin\theta'}{\sin\theta} = \frac{v}{v'} = \text{konst.} \qquad (A)$$

Med gitte spesifikke hastigheter for gitte medier er dette Snells brytningslov.

Merk at Newton i sin "partikkel modell" bruker større hastighet for et medium med større brytningsindeks mens for bølge modellen er det motsatt.

Eksperimentelt kan det vises at lyshastigheten minker når n øker (først vist av Foucault for vann i 1850).

Sammenholder vi Snells brytningslov med uttrykket vi utledet tidligere for relasjonen mellom innfallende vinkel og brytningsvinkel og bølgehastigheten får vi

$$\boxed{\frac{n_2}{n_1} = \frac{v_1}{v_2}}$$

Siden $v_i = \lambda_i f \quad (i = 1, 2)$ og f er uavhengig av mediet, må vi også ha

$$\frac{\lambda_2}{\lambda_1} = \frac{n_1}{n_2}.$$

(Dersom bølgelengden i vakuum er λ, blir den i et medium med n_i:

$$\boxed{\lambda_i = \frac{\lambda}{n_i}}$$

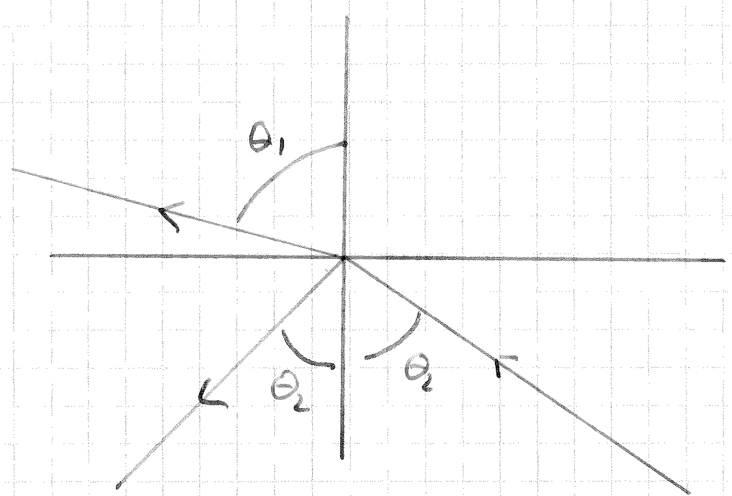
(F.eks. er bølgelengden i vann for rødt lys med $\lambda = 600$ nm (i vakuum) lik $\frac{600}{1{,}33}$ nm $= 450$ nm).

<u>Totalrefleksjon</u>

Vi har altså

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}.$$

Vi betrakter nå situasjonen at lyset kommer fra et medium med brytningsindeks n_2 til en grenseflate mot et medium med brytningsindeks $n_1 < n_2$:



θ_1 kan maksimalt bli $90°$.

Dersom ligningen $\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}$

gir at $\sin\theta_1 = \frac{n_2}{n_1}\sin\theta_2 > 1$,

betyr det at <u>alt lys blir</u>

<u>reflektert.</u>

Vi får <u>totalrefleksjon</u>.

Den vinkelen θ_2 som gir $\sin\theta_1 = 1$ kalles grensevinkelen eller den kritiske vinkel, θ_c, og er gitt ved

$$\boxed{\sin\theta_c = \frac{n_1}{n_2}}$$

<u>Eksempler:</u>

Grenseflate	n_2	n_1	θ_c
vann / luft	1.33	1.00	48.8°
glass / luft	1.5	1.00	41.8°
soyaolje / vann	1.5	1.33	62°

1) Prismekikkerter (gir nærmere 100% refleksjon – bedre enn de beste speil).

2) Optiske fibre. (Ofest glass med plastbelegg).

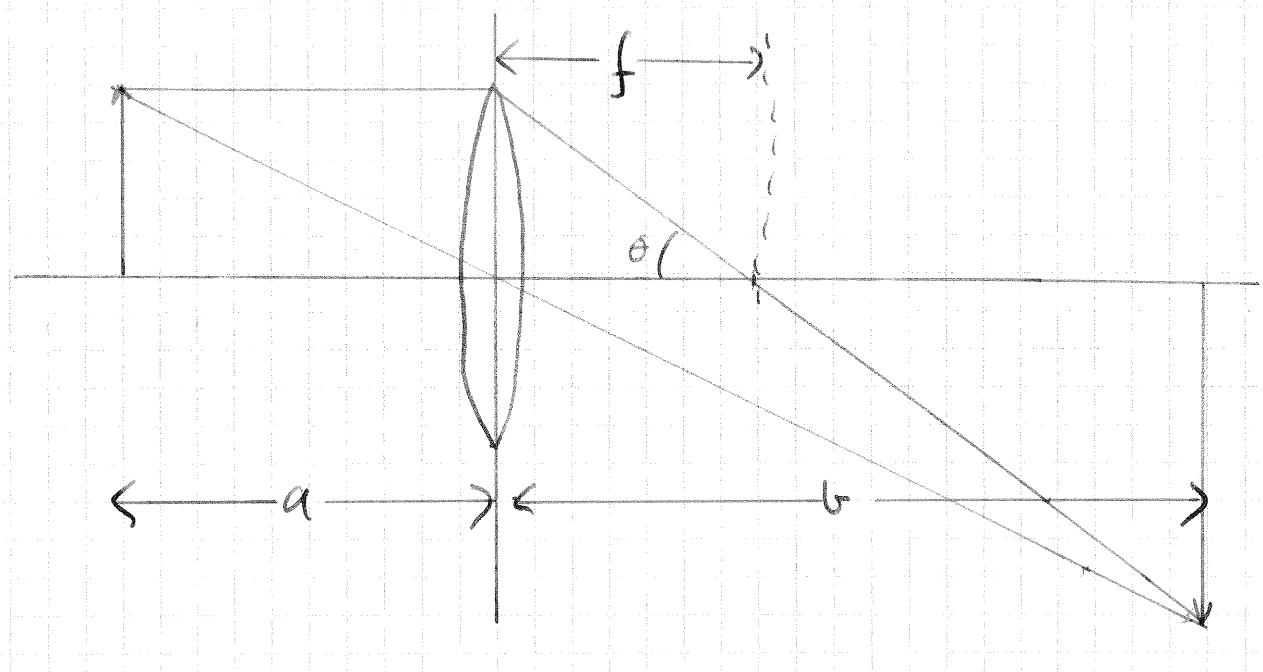
Kan bøyes kraftig uten at den kritiske vinkelen overskrides.

For overføring av bilder med optiske fibre må en benytte fiberbunter. Enkeltfibre må isoleres fra hverandre med stoff hvor $n < n_{glass}$.

Avbildning med samlelinse

(A&F 33.4 og 33.8).

Avbildning med samlelinse (konveks linse) er gjennomgått i 2FY – se f. eks. Øgrim et al. kap. 14.



Fra Øgrim et al. s. 315:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}.$$

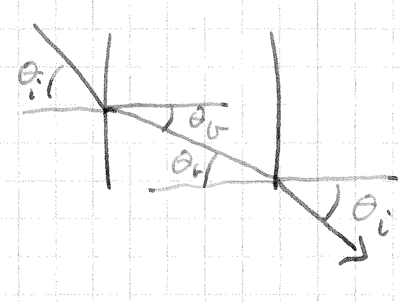
Forstørrelse (forminskning): $\frac{b}{a}$.

Konstruksjon av bilder.

1. En stråle som på objektsiden er parallell med aksen, får på bildesiden retning gjennom brennpunktet.

2. En stråle som på objektsiden passerer gjennom brennpunktet, blir på bildesiden parallell med aksen.

3. En stråle gjennom linsens midtpunkt endrer ikke retning.



Linsefeil (Aberrasjoner)

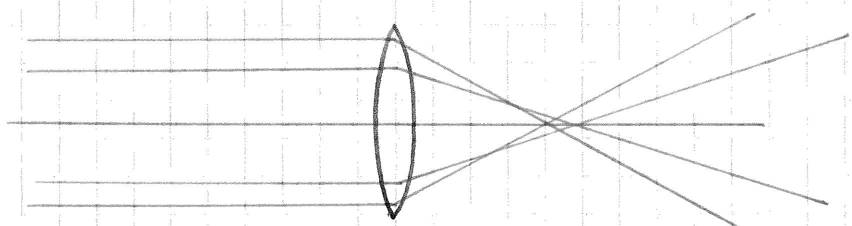
(Det at alle stråler parallelle til optisk akse brytes til ett brennpunkt er bare korrekt så lenge vi med god tilnærmelse kan sette

$$\sin\theta \simeq \theta .$$

Avviket for stråler som dette ikke gjelder tilstrekkelig godt for, kalles sfærisk aberrasjon

- sfærisk fordi man alltid får det ved brytning på kuleflater.

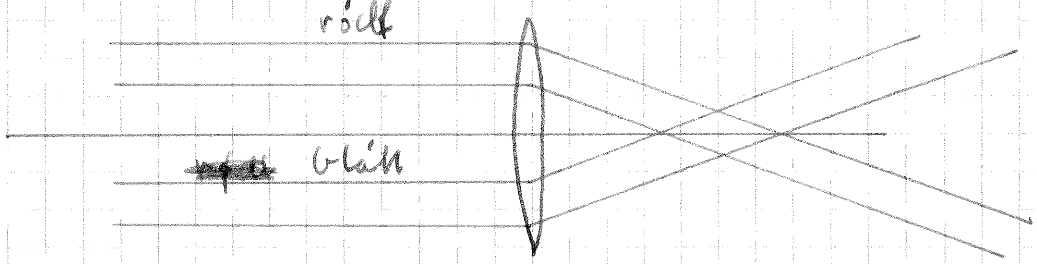
Linseflater er vanligvis del av kuleflater:



Merk at sfærisk aberrasjon her er tegnet svært overdrevet og at alle lysstrålene er tenkt å ha samme frekvens (og dermed farge).

"Avbildningsfeil" som skyldes at lys med forskjellig bølgelengde (frekvens, farge) har forskjellig brytningsindeks kalles

<u>kromatisk aberrasjon:</u>



(Blått lys brytes mer enn rødt lys.)

I tillegg vil bildet av store tredimensjonale objekter vise andre kompliserte forvrengninger.

Effekten av aberrasjoner kan gjøres vesentlig mindre ved hjelp av sammensatte linser.

Fargespekteret og dispersjon

(A&F 33.7)

Synlig lys har bølgelengde i luft i området 400 nm – 750 nm og frekvens i området $4 \cdot 10^{14}$ Hz – $7.5 \cdot 10^{14}$ Hz.

Forskjellige frekvenser tilsvarer forskjellige farger.

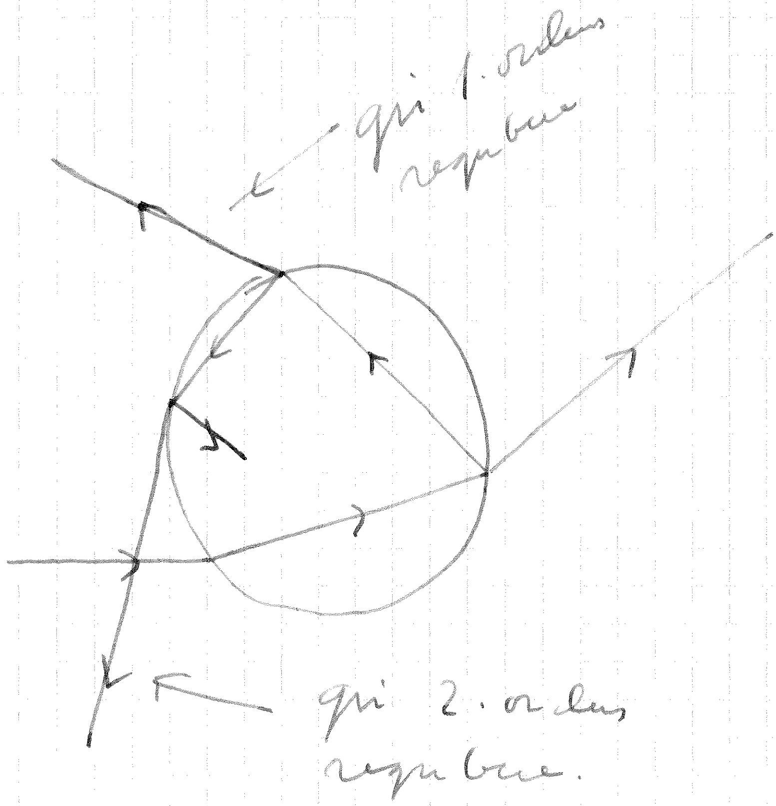
I mange stoff (f. eks. i glass og i vann) er brytningsindeksen avhengig av frekvensen.

Derfor blir bølgelengden og dermed fasehastigheten for lys i slike stoff frekvens-avhengig, dvs. vi har dispersjon.

P.g.a. dispersjon i glass kan et glassprisme separere hvitt lys i forskjellige farger.

P.g.a. dispersjon i vann vil også brytning (og refleksjon) i vanndråper adskille hvitt lys i forskjellige farger. Dette er opphav til regnbuer.

<u>Første ordens</u> regnbue består av lys reflektert én gang inne i regndråpen, <u>andre ordens</u> regnbue består av lys reflektert to ganger.



Se artikkel i Scientific American av H. M. Nussenzveig, April, 1977. (Ikke pensum!)

The Theory of the Rainbow

When sunlight is scattered by raindrops, why is it that colorful arcs appear in certain regions of the sky? Answering this subtle question has required all the resources of mathematical physics

by H. Moysés Nussenzveig

The rainbow is a bridge between the two cultures: poets and scientists alike have long been challenged to describe it. The scientific description is often supposed to be a simple problem in geometrical optics, a problem that was solved long ago and that holds interest today only as a historical exercise. This is not so; a satisfactory quantitative theory of the rainbow has been developed only in the past few years. Moreover, that theory involves much more than geometrical optics; it draws on all we know of the nature of light. Allowance must be made for wavelike properties, such as interference, diffraction and polarization, and for particlelike properties, such as the momentum carried by a beam of light.

Some of the most powerful tools of mathematical physics were devised explicitly to deal with the problem of the rainbow and with closely related problems. Indeed, the rainbow has served as a touchstone for testing theories of optics. With the more successful of those theories it is now possible to describe the rainbow mathematically, that is, to predict the distribution of light in the sky. The same methods can also be applied to related phenomena, such as the bright ring of color called the glory, and even to other kinds of rainbows, such as atomic and nuclear ones.

Scientific insight has not always been welcomed without reservations. Goethe wrote that Newton's analysis of the rainbow's colors would "cripple Nature's heart." A similar sentiment was expressed by Charles Lamb and John Keats; at a dinner party in 1817 they proposed a toast: "Newton's health, and confusion to mathematics." Yet the scientists who have contributed to the theory of the rainbow are by no means insensitive to the rainbow's beauty. In the words of Descartes: "The rainbow is such a remarkable marvel of Nature ... that I could hardly choose a more suitable example for the application of my method."

The single bright arc seen after a rain shower or in the spray of a waterfall is the primary rainbow. Certainly its most conspicuous feature is its splash of colors. These vary a good deal in brightness and distinctness, but they always follow the same sequence: violet is innermost, blending gradually with various shades of blue, green, yellow and orange, with red outermost.

Other features of the rainbow are fainter and indeed are not always present. Higher in the sky than the primary bow is the secondary one, in which the colors appear in reverse order, with red innermost and violet outermost. Careful observation reveals that the region between the two bows is considerably darker than the surrounding sky. Even when the secondary bow is not discernible, the primary bow can be seen to have a "lighted side" and a "dark side." The dark region has been given the name Alexander's dark band, after the Greek philosopher Alexander of Aphrodisias, who first described it in about A.D. 200.

Another feature that is only sometimes seen is a series of faint bands, usually pink and green alternately, on the inner side of the primary bow. (Even more rarely they may appear on the outer side of the secondary bow.) These "supernumerary arcs" are usually seen most clearly near the top of the bow. They are anything but conspicuous, but they have had a major influence on the development of theories of the rainbow.

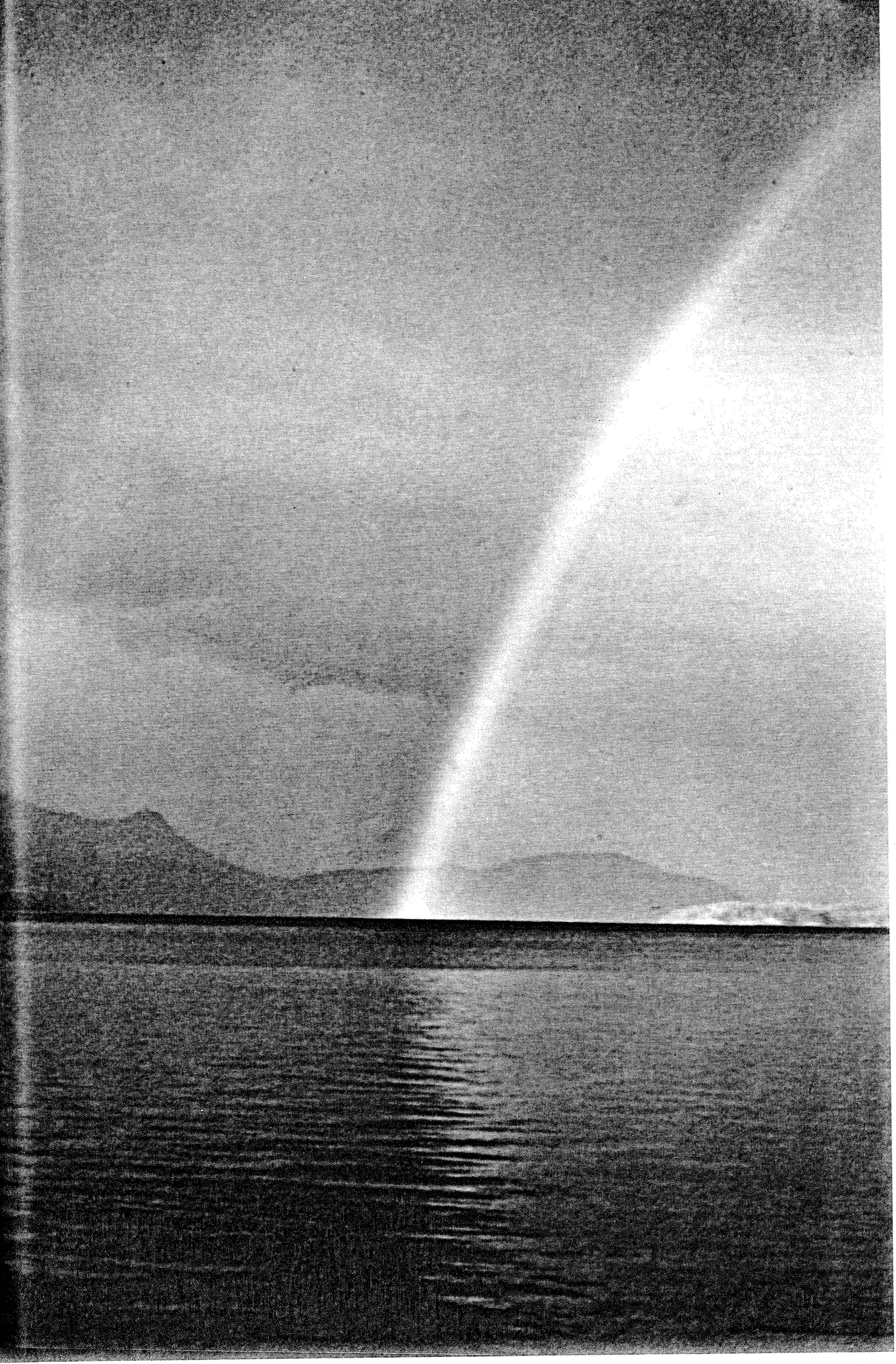
The first attempt to rationally explain the appearance of the rainbow was probably that of Aristotle. He proposed that the rainbow is actually an unusual kind of reflection of sunlight from clouds. The light is reflected at a fixed angle, giving rise to a circular cone of "rainbow rays." Aristotle thus explained correctly the circular shape of the bow and perceived that it is not a material object with a definite location in the sky but rather a set of directions along which light is strongly scattered into the eyes of the observer.

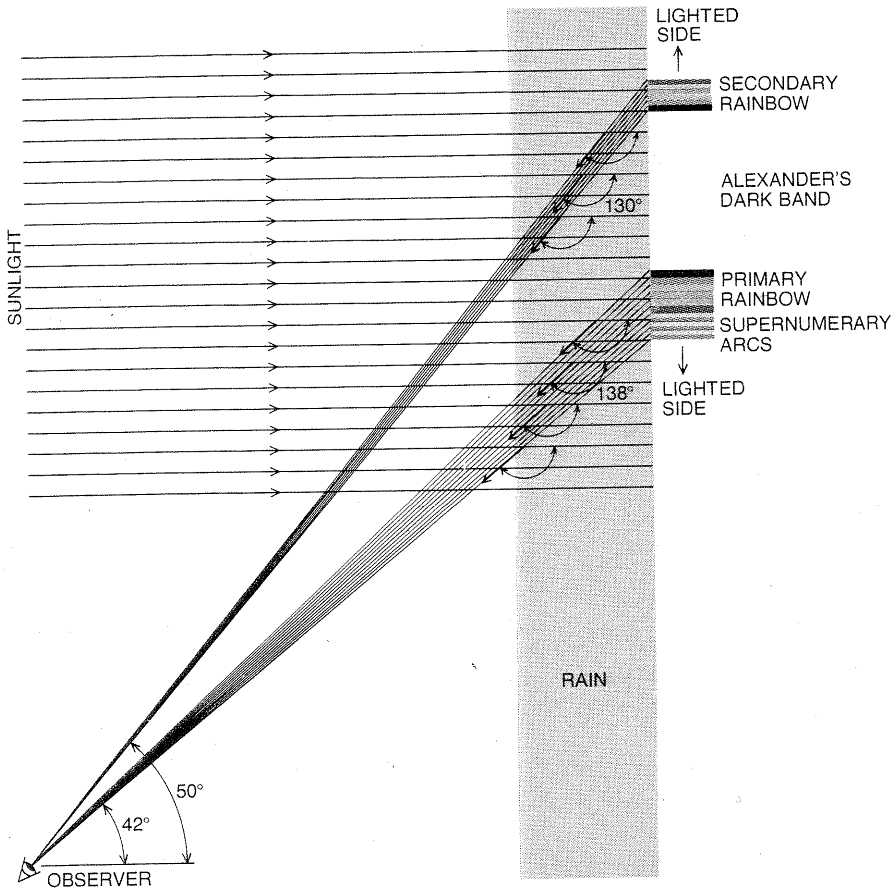
The angle formed by the rainbow rays and the incident sunlight was first measured in 1266 by Roger Bacon. He measured an angle of about 42 degrees; the secondary bow is about eight degrees higher in the sky. Today these angles are customarily measured from the opposite direction, so that we measure the total change in the direction of the sun's rays. The angle of the primary bow is therefore 180 minus 42, or 138, degrees; this is called the rainbow angle. The angle of the secondary bow is 130 degrees.

After Aristotle's conjecture some 17 centuries passed before further significant progress was made in the theory of the rainbow. In 1304 the German monk Theodoric of Freiberg rejected Aristotle's hypothesis that the rainbow results from collective reflection by the raindrops in a cloud. He suggested instead that each drop is individually capable of producing a rainbow. Moreover, he tested this conjecture in experiments with a magnified raindrop: a spherical flask filled with water. He was able to trace the path followed by the light rays that make up the rainbow.

Theodoric's findings remained largely unknown for three centuries, until they were independently rediscovered by Descartes, who employed the same method. Both Theodoric and Descartes showed that the rainbow is made up of rays that enter a droplet and are reflected once from the inner surface. The secondary bow consists of rays that have undergone two internal reflections. With each reflection some light is lost, which is the main reason the secondary bow is fainter than the primary one. Theodoric and Descartes also noted that along each direction within the angular

DOUBLE RAINBOW was photographed at Johnstone Strait in British Columbia. The bright, inner band is the primary bow; it is separated from the fainter secondary bow by a region, called Alexander's dark band, that is noticeably darker than the surrounding sky. Below the primary bow are a few faint stripes of pink and green; they are supernumerary arcs. The task of theory is to give a quantitative explanation for each of these features.





GEOMETRY OF THE RAINBOW is determined by the scattering angle: the total angle through which a ray of sunlight is bent by its passage through a raindrop. Rays are strongly scattered at angles of 138 degrees and 130 degrees, giving rise respectively to the primary and the secondary rainbows. Between those angles very little light is deflected; that is the region of Alexander's dark band. The optimum angles are slightly different for each wavelength of light, with the result that the colors are dispersed; note that the sequence of colors in the secondary bow is the reverse of that in the primary bow. There is no single plane in which the rainbow lies; the rainbow is merely the set of directions along which light is scattered toward the observer.

range corresponding to the rainbow only one color at a time could be seen in the light scattered by the globe. When the eye was moved to a new position so as to explore other scattering angles, the other spectral colors appeared, one by one. Theodoric and Descartes concluded that each of the colors in the rainbow comes to the eye from a different set of water droplets.

As Theodoric and Descartes realized, all the main features of the rainbow can be understood through a consideration of the light passing through a single droplet. The fundamental principles that determine the nature of the bow are those that govern the interaction of light with transparent media, namely reflection and refraction.

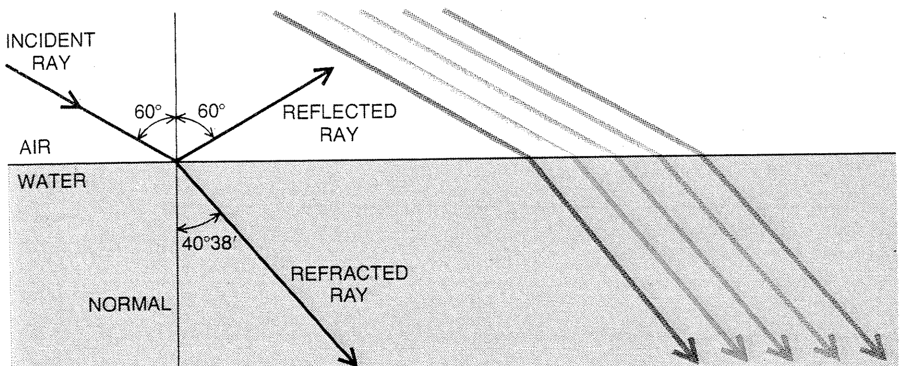
The law of reflection is the familiar and intuitively obvious principle that the angle of reflection must equal the angle of incidence. The law of refraction is somewhat more complicated. Whereas the path of a reflected ray is determined entirely by geometry, refraction also involves the properties of light and the properties of the medium.

The speed of light in a vacuum is invariant; indeed, it is one of the fundamental constants of nature. The speed of light in a material medium, on the other hand, is determined by the properties of the medium. The ratio of the speed of light in a vacuum to the speed in a substance is called the refractive index of that substance. For air the index is only slightly greater than 1; for water it is about 1.33.

A ray of light passing from air into water is retarded at the boundary; if it strikes the surface obliquely, the change in speed results in a change in direction. The sines of the angles of incidence and refraction are always in constant ratio to each other, and the ratio is equal to that between the refractive indexes for the two materials. This equality is called Snell's law, after Willebrord Snell, who formulated it in 1621.

A preliminary analysis of the rainbow can be obtained by applying the laws of reflection and refraction to the path of a ray through a droplet. Because the droplet is assumed to be spherical all directions are equivalent and there is only one significant variable: the displacement of the incident ray from an axis passing through the center of the droplet. That displacement is called the impact parameter. It ranges from zero, when the ray coincides with the central axis, to the radius of the droplet, when the ray is tangential.

At the surface of the droplet the incident ray is partially reflected, and this reflected light we shall identify as the scattered rays of Class 1. The remaining light is transmitted into the droplet (with a change in direction caused by refraction) and at the next surface is again partially transmitted (rays of Class 2) and partially reflected. At the next boundary the reflected ray is again split into reflected and transmitted components, and the process continues indefinitely. Thus the droplet gives rise to a series of scattered rays, usually with rapidly decreasing intensity. Rays of Class 1 represent direct reflection by the droplet and those of Class 2 are directly transmitted through it. Rays of Class 3 are those that escape the droplet after one internal reflection, and they make up the primary rainbow. The Class 4 rays, having undergone two internal re-



REFLECTION AND REFRACTION of light at boundaries between air and water are the basic events in the creation of a rainbow. In reflection the angle of incidence is equal to the angle of reflection. In refraction the angle of the transmitted ray is determined by the properties of the medium, as characterized by its refractive index. Light entering a medium with a higher index is bent toward the normal. Light of different wavelengths is refracted through slightly different angles; this dependence of the refractive index on color is called dispersion. Theories of the rainbow often deal separately with each monochromatic component of incident light.

flections, give rise to the secondary bow. Rainbows of higher order are formed by rays making more complicated passages, but they are not ordinarily visible.

For each class of scattered rays the scattering angle varies over a wide range of values as a function of the impact parameter. Since in sunlight the droplet is illuminated at all impact parameters simultaneously, light is scattered in virtually all directions. It is not difficult to find light paths through the droplet that contribute to the rainbow, but there are infinitely many other paths that direct the light elsewhere. Why, then, is the scattered intensity enhanced in the vicinity of the rainbow angle? It is a question Theodoric did not consider; an answer was first provided by Descartes.

By applying the laws of reflection and refraction at each point where a ray strikes an air-water boundary, Descartes painstakingly computed the paths of many rays incident at many impact parameters. The rays of Class 3 are of predominating importance. When the impact parameter is zero, these rays are scattered through an angle of 180 degrees, that is, they are backscattered toward the sun, having passed through the center of the droplet and been reflected from the far wall. As the impact parameter increases and the incident rays are displaced from the center of the droplet, the scattering angle decreases. Descartes found, however, that this trend does not continue as the impact parameter is increased to its maximum value, where the incident ray grazes the droplet at a tangent to its surface. Instead the scattering angle passes through a minimum when the impact parameter is about seven-eighths of the radius of the droplet, and thereafter it increases again. The scattering angle at the minimum is 138 degrees.

For rays of Class 4 the scattering angle is zero when the impact parameter is zero; in other words, the central ray is reflected twice, then continues in its original direction. As the impact parameter increases so does the scattering angle, but again the trend is eventually reversed, this time at 130 degrees. The Class 4 rays have a maximum scattering angle of 130 degrees, and as the impact parameter is further increased they bend back toward the forward scattering direction again.

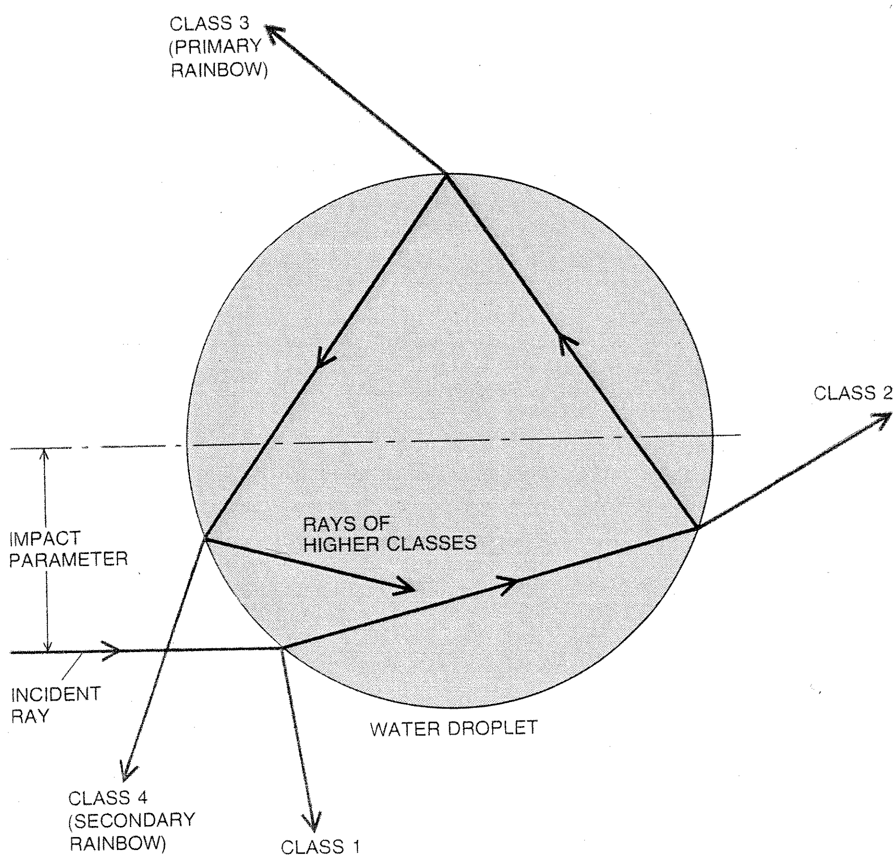
Because a droplet in sunlight is uniformly illuminated the impact parameters of the incident rays are uniformly distributed. The concentration of scattered light is therefore expected to be greatest where the scattering angle varies most slowly with changes in the impact parameter. In other words, the scattered light is brightest where it gathers together the incident rays from the largest range of impact parameters. The regions of minimum variation are those surrounding the maximum and minimum scattering angles, and so the special status of the primary and secondary rainbow angles is explained. Furthermore, since no rays of Class 3 or Class 4 are scattered into the angular region between 130 and 138 degrees, Alexander's dark band is also explained.

Descartes's theory can be seen more clearly by considering an imaginary population of droplets from which light is somehow scattered with uniform intensity in all directions. A sky filled with such droplets would be uniformly bright at all angles. In a sky filled with real water droplets the same total illumination is available, but it is redistributed. Most parts of the sky are dimmer than they would be with uniform scattering, but in the vicinity of the rainbow angle there is a bright arc, tapering off gradually on the lighted side and more sharply on the dark side. The secondary bow is a similar intensity highlight, except that it is narrower and all its features are dimmer. In the Cartesian theory the region between the bows is distinctly darker than the sky elsewhere; if only rays of Class 3 and Class 4 existed, it would be quite black.

The Cartesian rainbow is a remarkably simple phenomenon. Brightness is a function of the rate at which the scattering angle changes. That angle is itself determined by just two factors: the refractive index, which is assumed to be constant, and the impact parameter, which is assumed to be uniformly distributed. One factor that has no influence at all on the rainbow angle is size: the geometry of scattering is the same for small cloud droplets and for the large water-filled globes employed by Theodoric and Descartes.

So far we have ignored one of the most conspicuous features of the rainbow: its colors. They were explained, of course, by Newton, in his prism experiments of 1666. Those experiments demonstrated not only that white light is a mixture of colors but also that the refractive index is different for each color, the effect called dispersion. It follows that each color or wavelength of light must have its own rainbow angle; what we observe in nature is a collection of monochromatic rainbows, each one slightly displaced from the next.

From his measurements of the refractive index Newton calculated that the



PATH OF LIGHT through a droplet can be determined by applying the laws of geometrical optics. Each time the beam strikes the surface part of the light is reflected and part is refracted. Rays reflected directly from the surface are labeled rays of Class 1; those transmitted directly through the droplet are designated Class 2. The Class 3 rays emerge after one internal reflection; it is these that give rise to the primary rainbow. The secondary bow is made up of Class 4 rays, which have undergone two internal reflections. For rays of each class only one factor determines the value of the scattering angle. That factor is the impact parameter: the displacement of the incident ray from an axis that passes through the center of the droplet.

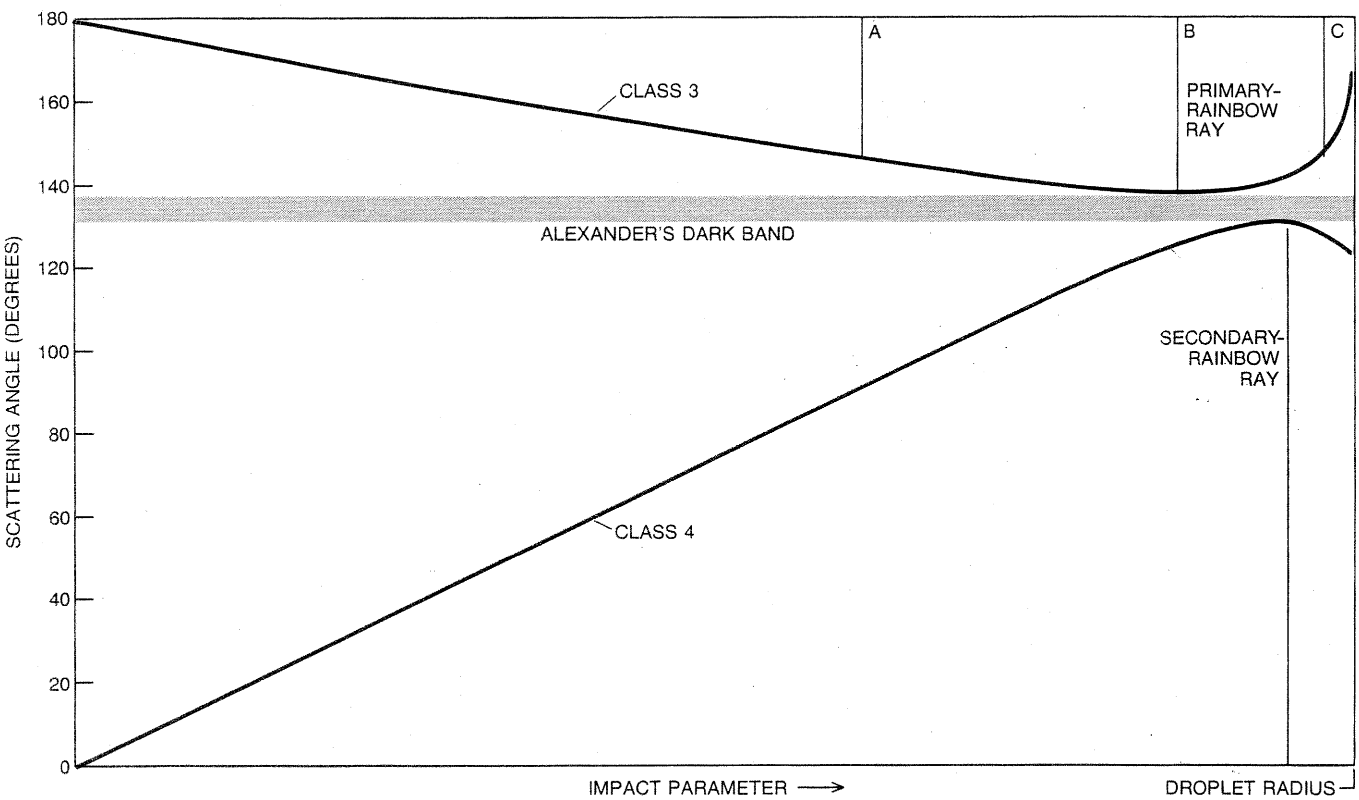
rainbow angle is 137 degrees 58 minutes for red light and 139 degrees 43 minutes for violet light. The difference between these angles is one degree 45 minutes, which would be the width of the rainbow if the rays of incident sunlight were exactly parallel. Allowing half a degree for the apparent diameter of the sun, Newton obtained a total width of two degrees 15 minutes for the primary bow. His own observations were in good agreement with this result.

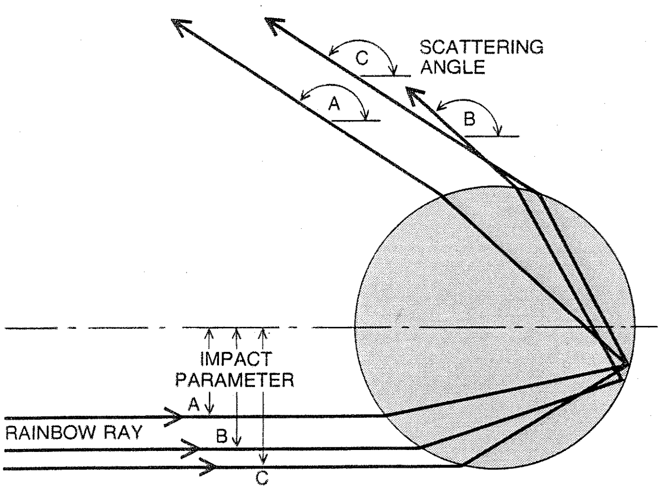
Descartes and Newton between them were able to account for all the more conspicuous features of the rainbow. They explained the existence of primary and secondary bows and of the dark band that separates them. They calculated the angular positions of these features and described the dispersion of the scattered light into a spectrum. All of this was accomplished with only geometrical optics. Their theory nevertheless had a major failing: it could not explain the supernumerary arcs. The understanding of these seemingly minor features requires a more sophisticated view of the nature of light.

The supernumerary arcs appear on the inner, or lighted, side of the primary bow. In this angular region two scattered rays of Class 3 emerge in the same direction; they arise from incident rays that have impact parameters on each side of the rainbow value. Thus at any given angle slightly greater than the rainbow angle the scattered light includes rays that have followed two different paths through the droplet. The rays emerge at different positions on the surface of the droplet, but they proceed in the same direction.

In the time of Descartes and Newton these two contributions to the scattered intensity could be handled only by simple addition. As a result the predicted intensity falls off smoothly with deviation from the rainbow angle, with no trace of supernumerary arcs. Actually the intensities of the two rays cannot be added because they are not independent sources of radiation.

The optical effect underlying the supernumerary arcs was discovered in 1803 by Thomas Young, who showed that light is capable of interference, a phenomenon that was already familiar from the study of water waves. In any medium the superposition of waves can lead either to reinforcement (crest on

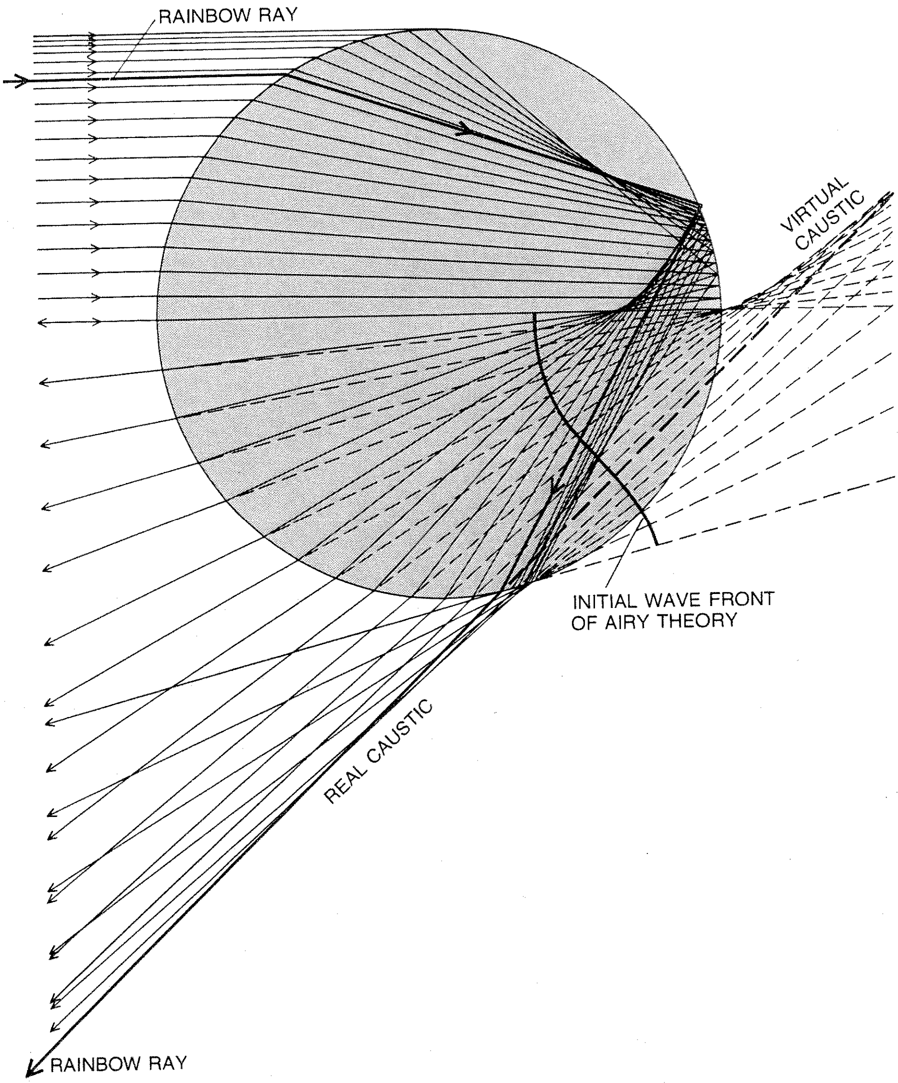




RAINBOW ANGLE can be seen to have a special significance when the scattering angle is considered as a function of the impact parameter. When the impact parameter is zero, the scattering angle for a ray of Class 3 is 180 degrees; the ray passes through the center of the droplet and is reflected by the far surface straight back at the sun. As the impact parameter increases, the scattering angle decreases, but eventually a minimum angle is reached. This ray of minimum deflection is the rainbow ray in the diagram at the left; rays with impact parameters on each side of it are scattered through larger angles. The minimum deflection is about 138 degrees, and the greatest concentration of scattered rays is to be found in the vicinity of this angle. The resulting enhancement in the intensity of the scattered light is perceived as the primary rainbow. The secondary bow is formed in a similar way, except that the scattering angle for the Class 4 rays of which it is composed increases to a maximum instead of decreasing to a minimum. The maximum lies at about 130 degrees. No rays of Class 3 or Class 4 can reach angles between 130 degrees and 138 degrees, explaining the existence of Alexander's dark band. At the left two Class 3 rays, with impact parameters on each side of the rainbow value, emerge at the same scattering angle. It is interference between rays such as these two that gives rise to the supernumerary arcs.

crest) or to cancellation (crest on trough). Young demonstrated the interference of light waves by passing a single beam of monochromatic light through two pinholes and observing the alternating bright and dark "fringes" produced. It was Young himself who pointed out the pertinence of his discovery to the supernumerary arcs of the rainbow. The two rays scattered in the same direction by a raindrop are strictly analogous to the light passing through the two pinholes in Young's experiment. At angles very close to the rainbow angle the two paths through the droplet differ only slightly, and so the two rays interfere constructively. As the angle increases, the two rays follow paths of substantially different length. When the difference equals half of the wavelength, the interference is completely destructive; at still greater angles the beams reinforce again. The result is a periodic variation in the intensity of the scattered light, a series of alternately bright and dark bands.

Because the scattering angles at which the interference happens to be constructive are determined by the difference between two path lengths, those angles are affected by the radius of the droplet. The pattern of the supernumerary arcs (in contrast to the rainbow angle) is therefore dependent on droplet size. In larger drops the difference in path length increases much more quickly with impact parameter than it does in small droplets. Hence the larger the droplets are, the narrower the angular separation between the supernumerary arcs is. The arcs can rarely be distinguished if the droplets are larger than about a millimeter in diameter. The overlapping of colors also tends to wash out the arcs. The size dependence of the supernumeraries explains why they are easier to see near the top of the bow: raindrops tend to grow larger as they fall.



CONFLUENCE OF RAYS **scattered by a droplet gives rise to caustics, or "burning curves." A caustic is the envelope of a ray system. Of special interest is the caustic of Class 3 rays, which has two branches, a real branch and a "virtual" one; the latter is formed when the rays are extended backward. When the rainbow ray is produced in both directions, it approaches the branches of this caustic. A theory of the rainbow based on the analysis of such a caustic was devised by George B. Airy. Having chosen an initial wave front—a surface perpendicular at all points to the rays of Class 3—Airy was able to determine the amplitude distribution in subsequent waves. A weakness of the theory is the need to guess the amplitudes of the initial waves.**

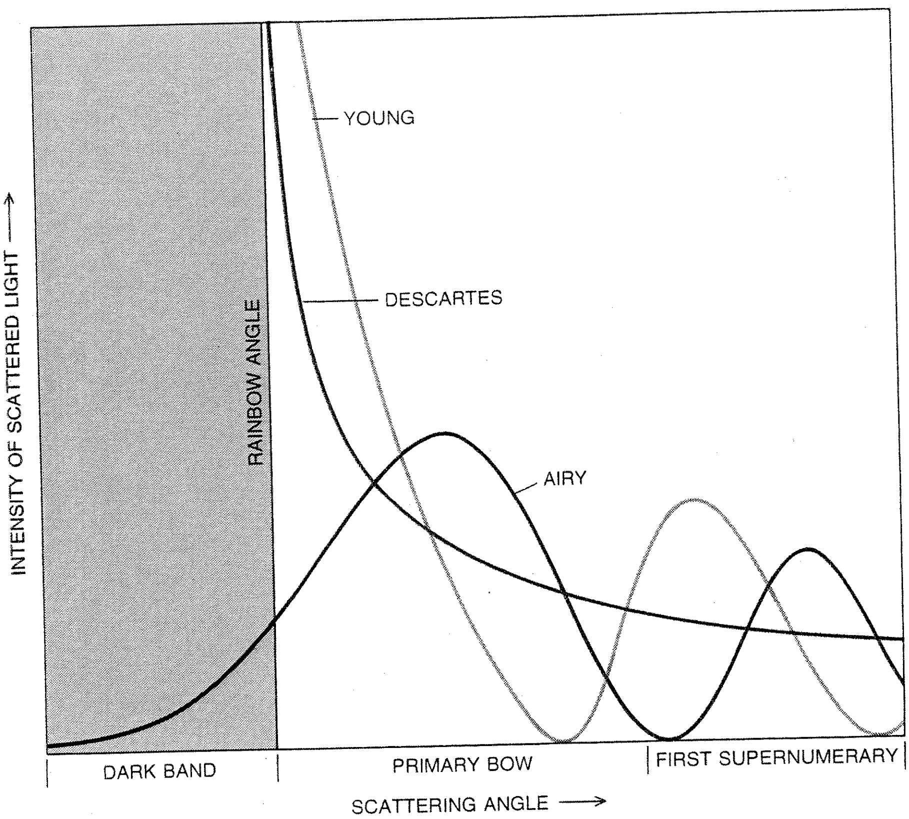
With Young's interference theory all the major features of the rainbow could be explained, at least in a qualitative and approximate way. What was lacking was a quantitative, mathematical theory capable of predicting the intensity of the scattered light as a function of droplet size and scattering angle.

Young's explanation of the supernumerary arcs was based on a wave theory of light. Paradoxically his predictions for the other side of the rainbow, for the region of Alexander's dark band, were inconsistent with such a theory. The interference theory, like the theories of Descartes and Newton, predicted complete darkness in this region, at least when only rays of Class 3 and Class 4 were considered. Such an abrupt transition, however, is not possible, because the wave theory of light requires that sharp boundaries between light and shadow be softened by diffraction. The most familiar manifestation of diffraction is the apparent bending of light or sound at the edge of an opaque obstacle. In the rainbow there is no real obstacle, but the boundary between the primary bow and the dark band should exhibit diffraction nonetheless. The treatment of diffraction is a subtle and difficult problem in mathematical physics, and the subsequent development of the theory of the rainbow was stimulated mainly by efforts to solve it.

In 1835 Richard Potter of the University of Cambridge pointed out that the crossing of various sets of light rays in a droplet gives rise to caustic curves. A caustic, or "burning curve," represents the envelope of a system of rays and is always associated with an intensity highlight. A familiar caustic is the bright cusp-shaped curve formed in a teacup when sunlight is reflected from its inner walls. Caustics, like the rainbow, generally have a lighted side and a dark side; intensity increases continuously up to the caustic, then drops abruptly.

Potter showed that the Descartes rainbow ray—the Class 3 ray of minimum scattering angle—can be regarded as a caustic. All the other transmitted rays of Class 3, when extended to infinity, approach the Descartes ray from the lighted side; there are no rays of this class on the dark side. Thus finding the intensity of the scattered light in a rainbow is similar to the problem of determining the intensity distribution in the neighborhood of a caustic.

In 1838 an attempt to determine that distribution was made by Potter's Cambridge colleague George B. Airy. His



PREDICTED INTENSITY as a function of scattering angle is compared for three early theories of the rainbow. In the geometric analysis of Descartes, intensity is infinite at the rainbow angle; it declines smoothly (without supernumerary arcs) on the lighted side and falls off abruptly to zero on the dark side. The theory of Thomas Young, which is based on the interference of light waves, predicts supernumerary arcs but retains the sharp transition from infinite to zero intensity. Airy's theory relocates the peaks in the intensity curve and for the first time provides (through diffraction) an explanation for gradual fading of the rainbow into shadow.

reasoning was based on a principle of wave propagation formulated in the 17th century by Christiaan Huygens and later elaborated by Augustin Jean Fresnel. This principle regards every point of a wave front as being a source of secondary spherical waves; the secondary waves define a new wave front and hence describe the propagation of the wave. It follows that if one knew the amplitudes of the waves over any one complete wave front, the amplitude distribution at any other point could be reconstructed. The entire rainbow could be described rigorously if we knew the amplitude distribution along a wave front in a single droplet. Unfortunately the amplitude distribution can seldom be determined; all one can usually do is make a reasonable guess for some chosen wave front in the hope that it will lead to a good approximation.

The starting wave front chosen by Airy is a surface inside the droplet, normal to all the rays of Class 3 and with an inflection point (a change in the sense of curvature) where it intersects the Descartes rainbow ray. The wave amplitudes along this wave front were estimated through standard assumptions in the theory of diffraction. Airy was then able to express the intensity of the scattered light in the rainbow region in terms of a new mathematical function, then known as the rainbow integral and today called the Airy function. The mathematical form of the Airy function will not concern us here; we shall concentrate instead on its physical meaning.

The intensity distribution predicted by the Airy function is analogous to the diffraction pattern appearing in the shadow of a straight edge. On the lighted side of the primary bow there are oscillations in intensity that correspond to the supernumerary arcs; the positions and widths of these peaks differ somewhat from those predicted by the Young interference theory. Another significant distinction of the Airy theory is that the maximum intensity of the rainbow falls at an angle somewhat greater than the Descartes minimum scattering angle. The Descartes and Young theories predict an infinite intensity at that angle (because of the caustic). The Airy theory does not reach an infinite intensity at any point, and at the Descartes rainbow ray the intensity predicted is less than half the maximum. Finally, diffraction effects appear on the dark side of the rainbow: instead of vanishing abruptly the intensity tapers away smoothly within Alexander's dark band.

Airy's calculations were for a monochromatic rainbow. In order to apply his method to a rainbow produced in sunlight one must superpose the Airy patterns generated by the various monochromatic components. To proceed further and describe the perceived image of the rainbow requires a theory of color vision.

The purity of the rainbow colors is determined by the extent to which the component monochromatic rainbows overlap; that in turn is determined by the droplet size. Uniformly large drops (with diameters on the order of a few millimeters) generally give bright rainbows with pure colors; with very small droplets (diameters of .01 millimeter or so) the overlap of colors is so great that the resulting light appears to be almost white.

An important property of light that we have so far ignored is its state of polarization. Light is a transverse wave, that is, one in which the oscillations are perpendicular to the direction of propagation. (Sound, on the other hand, is a longitudinal vibration.) The orientation of the transverse oscillation can be resolved into components along two mutually perpendicular axes. Any light ray can be described in terms of these two independent states of linear polarization. Sunlight is an incoherent mixture of the two in equal proportions; it is often said to be randomly polarized or simply unpolarized. Reflection can alter its state of polarization, and in that fact lies the importance of polarization to the analysis of the rainbow.

Let us consider the reflection of a light ray traveling inside a water droplet when it reaches the boundary of the droplet. The plane of reflection, the plane that contains both the incident and the reflected rays, provides a convenient geometric reference. The polarization states of the incident light can be defined as being parallel to that plane and perpendicular to it. For both polarizations the reflectivity of the surface is slight at angles of incidence near the perpendicular, and it rises very steeply near a critical angle whose value is determined by the index of refraction. Beyond that critical angle the ray is totally reflected, regardless of polarization. At intermediate angles, however, reflectivity depends on polarization. As the angle of incidence becomes shallower a steadily larger portion of the perpendicularly polarized component is reflected. For the parallel component, on the other hand, reflectivity falls before it begins to increase. At one angle in particular, reflectivity for the parallel-polarized wave vanishes entirely; that wave is totally transmitted. Hence for sunlight incident at that angle the internally reflected ray is completely polarized perpendicular to the plane of reflection. The angle is called Brewster's angle, after David Brewster, who discussed its significance in 1815.

Light from the rainbow is almost completely polarized, as can be seen by looking at a rainbow through Polaroid sunglasses and rotating the lenses around the line of sight. The strong polarization results from a remarkable coincidence: the internal angle of incidence for the rainbow ray is very close to Brewster's angle. Most of the parallel component escapes in the transmitted rays of Class 2, leaving a preponderance of perpendicular rays in the rainbow.

With the understanding that both matter and radiation can behave as waves, the theory of the rainbow has been enlarged in scope. It must now encompass new, invisible rainbows produced in atomic and nuclear scattering.

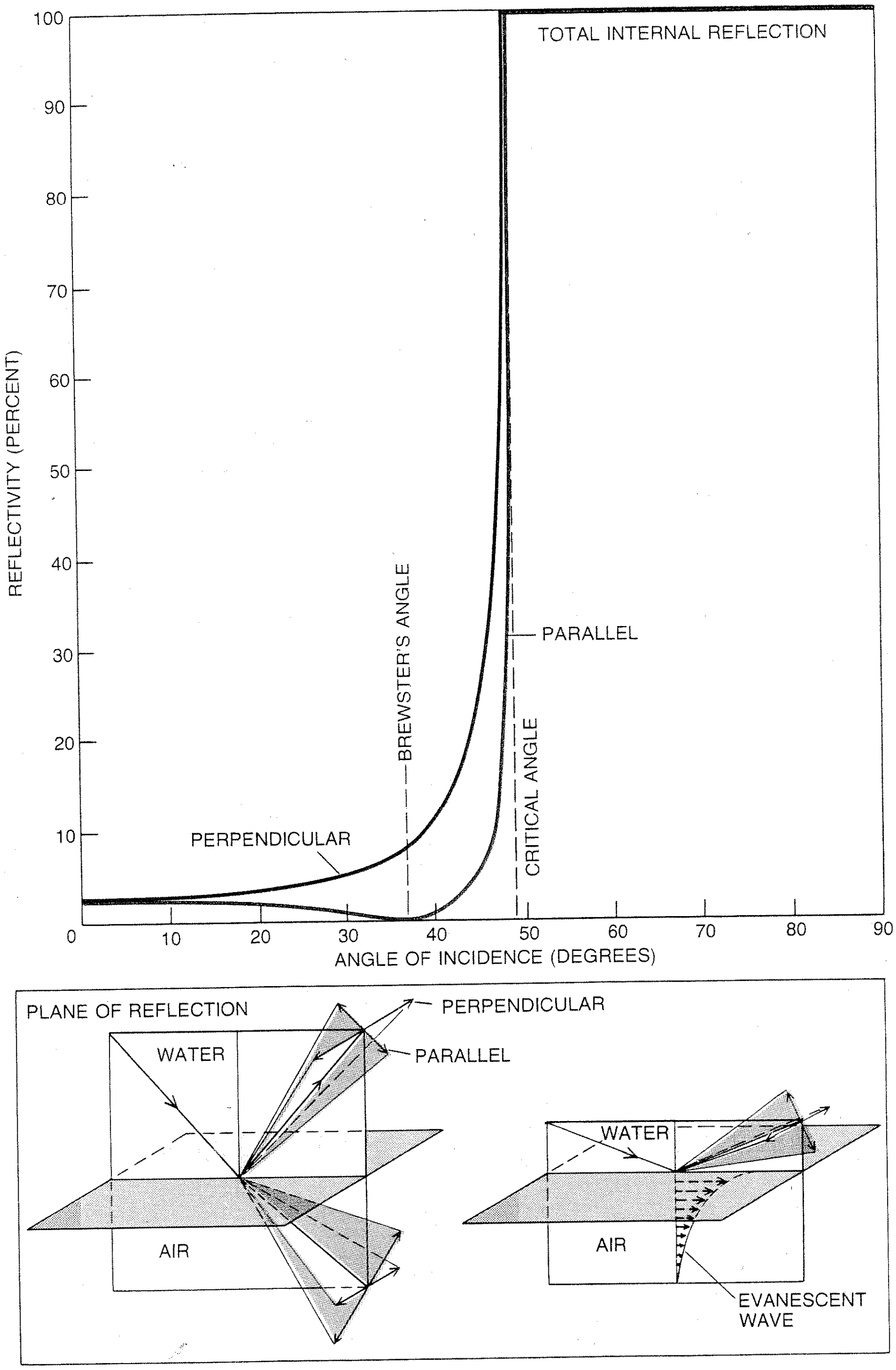
An analogy between geometrical optics and classical particle mechanics had already been perceived in 1831 by William Rowan Hamilton, the Irish mathematician. The analogues of rays in geometrical optics are particle trajectories, and the bending of a light ray on entering a medium with a different refractive index corresponds to the deflection of a moving particle under the action of a force. Particle-scattering analogues exist for many effects in optics, including the rainbow.

Consider a collision between two atoms in a gas. As the atoms approach from a large initial separation, they are at first subject to a steadily increasing attraction. At closer range, however, the electron shells of the atoms begin to interpenetrate and the attractive force diminishes. At very close range it becomes an increasingly strong repulsion.

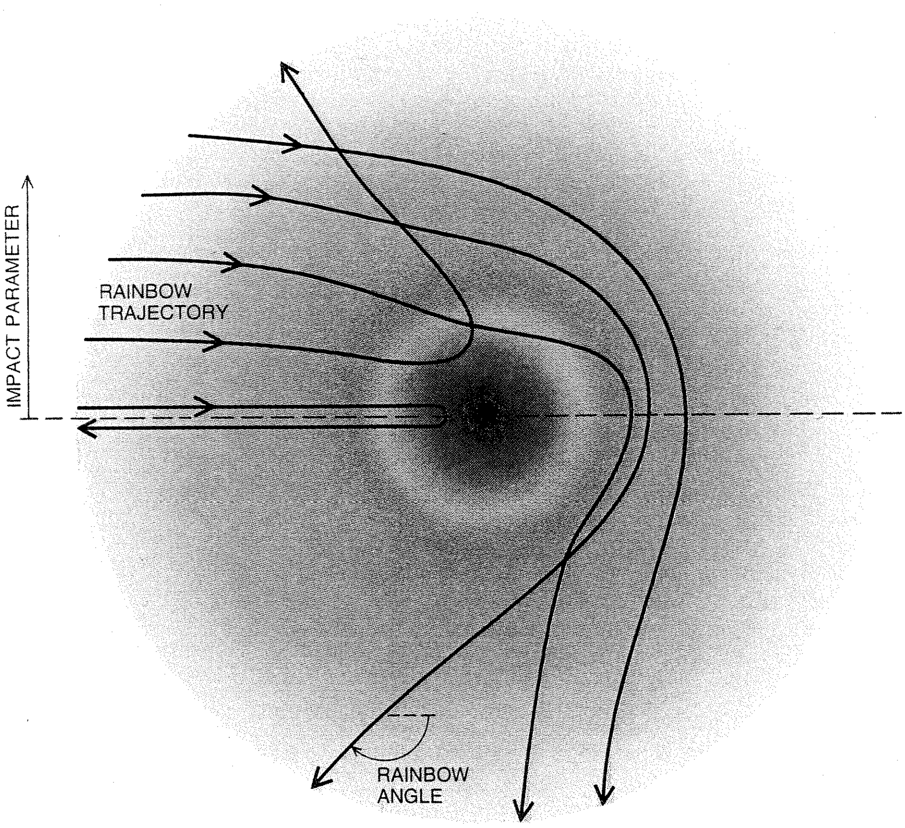
As in the optical experiment, the atomic scattering can be analyzed by tracing the paths of the atoms as a function of the impact parameter. Because the forces vary gradually and continuously, the atoms follow curved trajectories instead of changing direction suddenly, as at the boundary between media of differing refractive index. Even though some of the trajectories are rather complicated, each impact parameter corresponds to a single deflection angle; moreover, there is one trajectory that represents a local maximum angular deflection. That trajectory turns out to be the one that makes the most effective use of the attractive interaction between atoms. A strong concentration of scattered particles is expected near this angle; it is the rainbow angle for the interacting atoms.

A wave-mechanical treatment of the atomic and nuclear rainbows was formulated in 1959 by Kenneth W. Ford of Brandeis University and John A. Wheeler of Princeton University. Interference between trajectories emerging in the same direction gives rise to supernumerary peaks in intensity. A particle-scattering analogue of Airy's theory has also been derived.

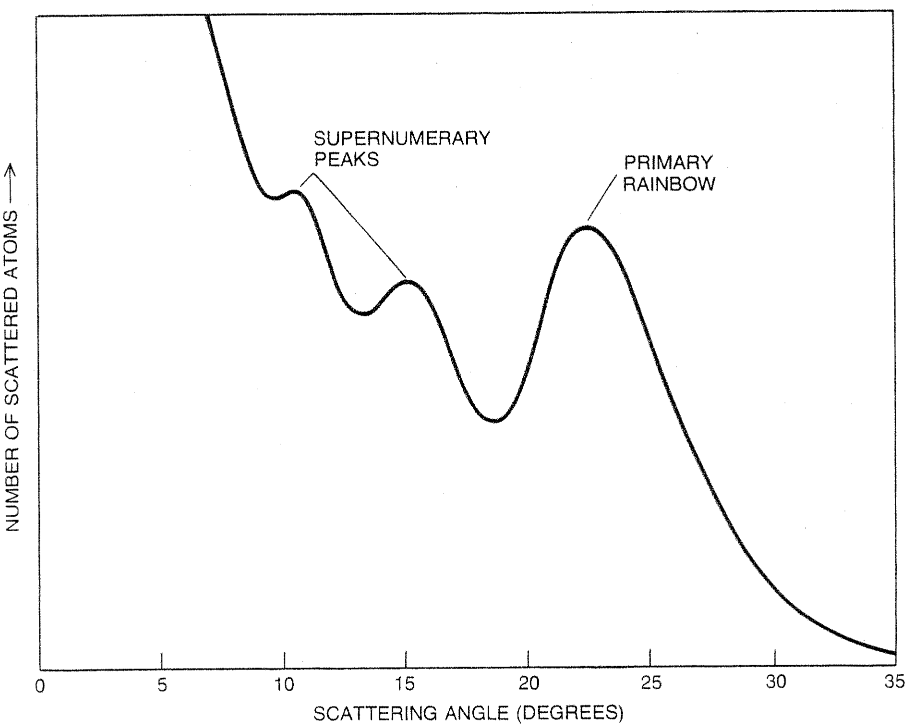
An atomic rainbow was first observed in 1964, by E. Hundhausen and H. Pauly of the University of Bonn, in the scattering of sodium atoms by mercury atoms. The main rainbow peak and two supernumeraries were detected; in more recent experiments oscillations on an even finer scale have been observed. The rainbows measured in these experiments carry information about the interatomic forces. Just as the optical rain-



POLARIZATION OF THE RAINBOW results from differential reflection. An incident ray can be resolved into two components polarized parallel to and perpendicular to the plane of reflection. For a ray approaching an air-water boundary from inside a droplet the reflectivity of the surface depends on the angle of incidence. Beyond a critical angle both parallel and perpendicular components are totally reflected, although some light travels parallel to the surface as an "evanescent wave." At lesser angles the perpendicular component is reflected more efficiently than the parallel one, and at one angle in particular, Brewster's angle, parallel-polarized light is completely transmitted. The angle of internal reflection for the rainbow ray falls near Brewster's angle. As a result light from the rainbow has a strong perpendicular polarization.



SCATTERING OF ATOMS BY ATOMS creates a particulate rainbow. The role played in optical scattering by the refractive index is played here by interatomic forces. The principal difference is that the forces vary smoothly and continuously, so that the atoms follow curved trajectories. As one atom approaches another the force between them is initially a steadily growing attraction (*colored shading*), but at close range it becomes strongly repulsive (*gray shading*). A local maximum in the scattering angle corresponds to the optical rainbow angle. It is the angle made by the trajectory most effective in using the attractive part of the potential.



ATOMIC RAINBOW was detected by E. Hundhausen and H. Pauly of the University of Bonn in the scattering of sodium atoms by mercury atoms. The oscillations in the number of scattered atoms detected correspond to a primary rainbow and to two supernumerary peaks. A rainbow of this kind embodies information about the strength and range of the interatomic forces.

bow angle depends solely on the refractive index, so the atomic rainbow angle is determined by the strength of the attractive part of the interaction. Similarly, the positions of the supernumerary peaks are size-dependent, and they provide information about the range of the interaction. Observations of the same kind have now been made in the scattering of atomic nuclei.

The Airy theory of the rainbow has had many satisfying successes, but it contains one disturbing uncertainty: the need to guess the amplitude distribution along the chosen initial wave front. The assumptions employed in making that guess are plausible only for rather large raindrops. In this context size is best expressed in terms of a "size parameter," defined as the ratio of a droplet's circumference to the wavelength of the light. The size parameter varies from about 100 in fog or mist to several thousand for large raindrops. Airy's approximation is plausible only for drops with a size parameter greater than about 5,000.

It is ironic that a problem as intractable as the rainbow actually has an exact solution, and one that has been known for many years. As soon as the electromagnetic theory of light was proposed by James Clerk Maxwell about a century ago, it became possible to give a precise mathematical formulation of the optical rainbow problem. What is needed is a computation of the scattering of an electromagnetic plane wave by a homogeneous sphere. The solution to a similar but slightly easier problem, the scattering of sound waves by a sphere, was discussed by several investigators, notably Lord Rayleigh, in the 19th century. The solution they obtained consisted of an infinite series of terms, called partial waves. A solution of the same form was found for the electromagnetic problem in 1908 by Gustav Mie and Peter J. W. Debye.

Given the existence of an exact solution to the scattering problem, it might seem an easy matter to determine all its features, including the precise character of the rainbow. The problem, of course, is the need to sum the series of partial waves, each term of which is a rather complicated function. The series can be truncated to give an approximate solution, but this procedure is practical only in some cases. The number of terms that must be retained is of the same order of magnitude as the size parameter. The partial-wave series is therefore eminently suited to the treatment of Rayleigh scattering, which is responsible for the blue of the sky; in that case the scattering particles are molecules and are much smaller than the wavelength, so that one term of the series is enough. For the rainbow problem size parameters up to several thousand must be considered.

A good approximation to the solution by the partial-wave method would require evaluating the sum of several thousand complicated terms. Computers have been applied to the task, but the results are rapidly varying functions of the size parameter and the scattering angle, so that the labor and cost quickly become prohibitive. Besides, a computer can only calculate numerical solutions; it offers no insight into the physics of the rainbow. We are thus in the tantalizing situation of knowing a form of the exact solution and yet being unable to extract from it an understanding of the phenomena it describes.

The first steps toward the resolution of this paradox were taken in the early years of the 20th century by the mathematicians Henri Poincaré and G. N. Watson. They found a method for transforming the partial-wave series, which converges only very slowly onto a stable value, into a rapidly convergent expression. The technique has come to be known as the Watson transformation or as the complex-angular-momentum method.

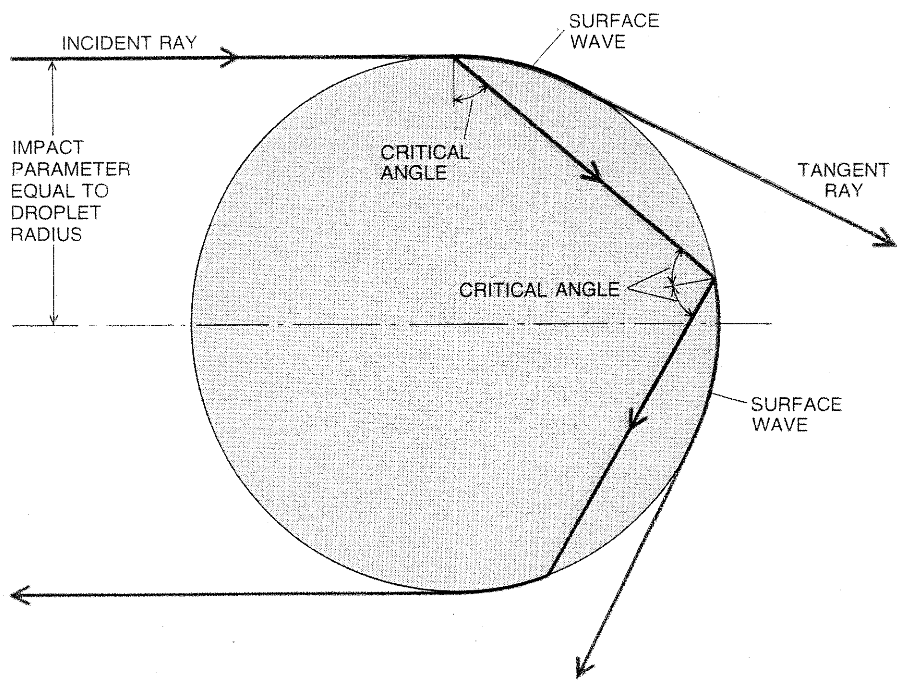
It is not particularly hard to see why angular momentum is involved in the rainbow problem, although it is less obvious why "complex" values of the angular momentum need to be considered. The explanation is simplest in a corpuscular theory of light, in which a beam of light is regarded as a stream of the particles called photons. Even though the photon has no mass, it does transport energy and momentum in inverse proportion to the wavelength of the corresponding light wave. When a photon strikes a water droplet with some impact parameter greater than zero, the photon carries an angular momentum equal to the product of its linear momentum and the impact parameter. As the photon undergoes a series of internal reflections, it is effectively orbiting the center of the droplet. Actually quantum mechanics places additional constraints on this process. On the one hand it requires that the angular momentum assume only certain discrete values; on the other it denies that the impact parameter can be precisely determined. Each discrete value of angular momentum corresponds to one term in the partial-wave series.

In order to perform the Watson transformation, values of the angular momentum that are conventionally regarded as being "unphysical" must be introduced. For one thing the angular momentum must be allowed to vary continuously, instead of in quantized units; more important, it must be allowed to range over the complex numbers: those that include both a real component and an imaginary one, containing some multiple of the square root of -1. The plane defined by these two components is referred to as the complex-angular-momentum plane.

Much is gained in return for the mathematical abstractions of the complex-angular-momentum method. In particular, after going over to the complex-angular-momentum plane through the Watson transformation, the contributions to the partial-wave series can be redistributed. Instead of a great many terms, one can work with just a few points called poles and saddle points in the complex-angular-momentum plane. In recent years the poles have attracted great theoretical interest in the physics of elementary particles. In that context they are usually called Regge poles, after the Italian physicist Tullio Regge.

Both poles and saddle points have physical interpretations in the rainbow problem. Contributions from real saddle points are associated with the ordinary, real light rays we have been considering throughout this article. What about complex saddle points? Imaginary or complex numbers are ordinarily regarded as being unphysical solutions to an equation, but they are not meaningless solutions. In descriptions of wave propagation imaginary components are usually associated with the damping of the wave amplitude. For example, in the total internal reflection of a light ray at a water-air boundary a light wave does go "through the looking glass." Its amplitude is rapidly damped, however, so that the intensity becomes negligible within a depth on the order of a single wavelength. Such a wave does not propagate into the air; instead it becomes attached to the interface between the water and the air, traveling along the surface; it is called an evanescent wave. The mathematical description of the evanescent wave involves the imaginary components of a solution. The effect called quantum-mechanical tunneling, in which a particle passes through a potential barrier without climbing over it, has a similar mathematical basis. "Complex rays" also appear on the shadow side of a caustic, where they describe the damped amplitude of the diffracted light waves.

Regge-pole contributions to the transformed partial-wave series are associated with surface waves of another kind. These waves are excited by incident rays that strike the sphere tangentially. Once such a wave is launched, it travels around the sphere, but it is continually damped because it sheds radiation tangentially, like a garden sprinkler. At each point along the wave's circumferential path it also penetrates the sphere at the critical angle for total internal reflection, reemerging as a surface wave after taking one or more such shortcuts. It is interesting to note that Johannes Kepler conjectured in 1584 that "pin-



COMPLEX-ANGULAR-MOMENTUM theory of the rainbow begins with the observation that a photon, or quantum of light, incident on a droplet at some impact parameter (which cannot be exactly defined) carries angular momentum. In the theory, components of that angular momentum are extended to complex values, that is, values containing the square root of -1. The consequences of this procedure can be illustrated by the example of a ray striking a droplet tangentially. The ray stimulates surface waves, which travel around the droplet and continuously shed radiation. The ray can also penetrate the droplet at the critical angle for total internal reflection, emerging either to form another surface wave or to repeat the shortcut.

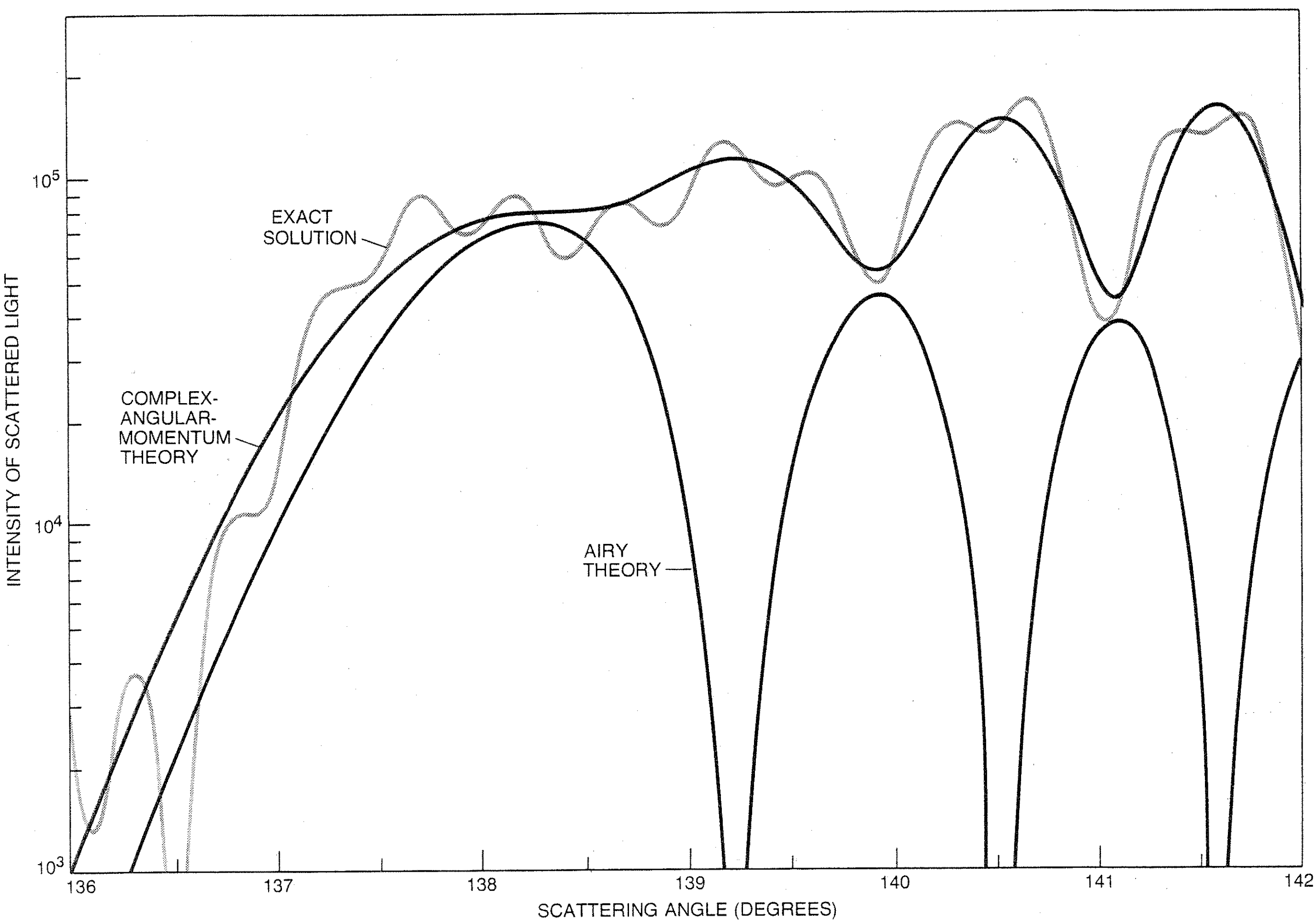
wheel" rays of this kind might be responsible for the rainbow, but he abandoned the idea because it did not lead to the correct rainbow angle.

In 1937 the Dutch physicists Balthus Van der Pol and H. Bremmer applied Watson's transformation to the rainbow problem, but they were able to show only that Airy's approximation could be obtained as a limiting case. In 1965 I developed an improved version of Watson's method, and I applied it to the rainbow problem in 1969 with somewhat greater success.

In the simple Cartesian analysis we saw that on the lighted side of the rainbow there are two rays emerging in the same direction; at the rainbow angle these coalesce into the single Descartes ray of minimum deflection and on the shadow side they vanish. In the complex-angular-momentum plane, as I have mentioned, each geometric ray corresponds to a real saddle point. Hence in mathematical terms a rainbow is merely the collision of two saddle points in the complex-angular-momentum plane. In the shadow region beyond the rainbow angle the saddle points do not simply disappear; they become complex, that is, they develop imaginary parts. The diffracted light in Alexander's dark band arises from a complex saddle point. It is an example of a "complex ray" on the shadow side of a caustic curve.

It should be noted that the adoption of the complex-angular-momentum method does not imply that earlier solutions to the rainbow problem were wrong. Descartes's explanation of the primary bow as the ray of minimum deflection is by no means invalid, and the supernumerary arcs can still be regarded as a product of interference, as Young proposed. The complex-angular-momentum method simply gives a more comprehensive accounting of the paths available to a photon in the rainbow region of the sky, and it thereby achieves more accurate results.

In 1975 Vijay Khare of the University of Rochester made a detailed comparison of three theories of the rainbow: the Airy approximation, the "exact" solution, obtained by a computer summation of the partial-wave series, and the rainbow terms in the complex-angular-momentum method, associated with the collision of two saddle points. For the dominant, perpendicular polarization the Airy theory requires only small corrections within the primary bow, and its errors become appreciable only in the region of the supernumerary arcs. For the scattered rays polarized parallel to the scattering plane, however, Airy's approximation fails badly. For the supernumerary arcs the exact solution shows minima where the Airy theory has maximum intensity, and vice versa. This serious failure is an indirect result of the near coincidence between the angle of internal reflection for the rainbow rays and Brewster's angle. At Brewster's angle the amplitude of the reflected ray changes sign, a change the Airy theory does not take into account. As a result of the change in sign the interference along directions corresponding to the peaks in



QUANTITATIVE THEORIES of the rainbow predict the intensity of the scattered light as a function of the scattering angle and also with respect to droplet size and polarization. Here the predictions of three theories are presented for parallel-polarized light scattered by droplets with a circumference equal to 1,500 wavelengths of the light. One curve represents the "exact" solution to the rainbow problem, derived from James Clerk Maxwell's equations describing electromagnetic radiation. The exact solution is the sum of an infinite series of terms, approximated here by adding up more than 1,500 complicated terms for each point employed in plotting the curve. The Airy theory is clearly in disagreement with the exact solution, particularly in the angular region of the supernumerary arcs. There the exact solution shows troughs at the positions of Airy's peaks. The results obtained by the complex-angular-momentum method, on the other hand, correspond closely to the exact solution, failing only to reproduce small, high-frequency oscillations. These fluctuations are associated with another optical phenomenon in the atmosphere, the glory, which is also explained by the complex-angular-momentum theory.

the Airy solutions is destructive instead of constructive.

In terms of large-scale features, such as the primary bow, the supernumerary arcs and the dark-side diffraction pattern, the complex-angular-momentum result agrees quite closely with the exact solution. Smaller-scale fluctuations in the exact intensity curve are not reproduced as well by the rainbow terms in the complex-angular-momentum method. On the other hand, the exact solution, for a typical size parameter of 1,500, requires the summation of more than 1,500 complicated terms; the complex-angular-momentum curve is obtained from only a few much simpler terms.

The small residual fluctuations in the exact intensity curve arise from higher-order internal reflections: rays belonging to classes higher than Class 3 or Class 4. They are of little importance for the primary bow, but at larger scattering angles their contribution increases and near the backward direction it becomes dominant. There these rays are responsible for another fascinating meteorological display: the glory [see "The Glory," by Howard C. Bryant and Nelson Jarmie; SCIENTIFIC AMERICAN, July, 1974].

The glory appears as a halo of spectral colors surrounding the shadow an observer casts on clouds or fog; it is most commonly seen from an airplane flying above clouds. It can also be explained through the complex-angular-momentum theory, but the explanation is more complicated than that for the rainbow. One set of contributions to the glory comes from the surface waves described by Regge poles that are associated with the tangential rays of Kepler's pinwheel type. Multiple internal reflections that happen to produce closed, star-shaped polygons play an important role, leading to resonances, or enhancements in intensity. Such geometric coincidences are very much in the spirit of Kepler's theories.

A second important set of contributions, demonstrated by Khare, is from the shadow side of higher-order rainbows that appear near the backward direction. These contributions represent the effect of complex rays. The 10th-order rainbow, formed only a few degrees away from the backward direction, is particularly effective.

For the higher-order rainbows Airy's theory would give incorrect results for both polarizations, and so the complex-angular-momentum theory must be employed. One might thus say the glory is formed in part from the shadow of a rainbow. It is gratifying to discover in the elegant but seemingly abstract theory of complex angular momentum an explanation for these two natural phenomena, and to find there an unexpected link between them.