

Geometric visualization of the Brewster angle from dielectric–magnetic interface

Ari Sihvola*

*Helsinki University of Technology,
Electromagnetics Laboratory*

P.O. Box 3000, Espoo, FIN-02015, Finland †

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A geometric visualization is presented for the Brewster angle for a plane wave reflecting from an interface. The surface is assumed isotropic but it is allowed to display both dielectric and magnetic susceptibility, and hence the Brewster (polarizing) angle can attain any value between 0 and 90 degrees, and can exist for both parallel and perpendicular polarizations. The geometric construction (a tetrahedron) is spanned by the basic material parameters of the surface. The Brewster angle appears in one of the faces of the tetrahedron.

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I. INTRODUCTION

Very much physics is sometimes contained in simple and basic results of optics and electromagnetics. In this paper I shall focus on the character of electromagnetic waves reflected from a planar surface. As is well known, many everyday light phenomena that we can observe with plain eyes [1] can be justified and explained with basic wave theory which is being taught to freshmen in physics and engineering schools. As examples in optics we could mention the glare on road surfaces on a sunny day which can be reduced by use of Polaroid sun glasses, or the way the images reflected from a water surface differ from those that are directly observed.

The polarization state of light changes in refraction and reflection processes. Since our eyes are not capable of sensing polarization, and natural light very often is rather unpolarized, the subtleties of the outdoor images, as they appear to us, may only be present in very indirect ways. But one especially interesting phenomenon in this respect is the possibility of light to become fully polarized in reflection. This happens when light impinges on a surface in a certain direction, from the Brewster angle. In the following, let us concentrate on the dependence of Brewster angle on the fundamental material parameters. In particular, the emphasis shall be on the way how the Brewster angle can be visualized in a geometrical way which contains pedagogical and physical insight.

In the following, the materials to be analyzed are assumed isotropic and lossless. However, in one respect the analysis is more general than that encountered in basic textbooks in optics which often restrict the treatment to non-magnetic media: here also magnetic permeability is taken as a material parameter that can vary. Presently in many engineering applications, composite materials research, and nanotechnology, great interest is in the magnetic properties of matter, which gives motivation to al-

low magnetic contrasts in the studies of canonical problems.

Hence, if both electric and magnetic responses are present, the material from which the wave reflects is characterized by two parameters, the relative permittivity and permeability ϵ and μ . These are assumed in the present paper to be real and positive [11]. But to ease the analysis, instead of using these parameters, it appears more convenient to apply the refractive index n and relative impedance η of the material:

$$n = \sqrt{\epsilon\mu}, \quad \eta = \sqrt{\mu/\epsilon} \quad (1)$$

Obviously the inverse relations are $\epsilon = n/\eta$ and $\mu = n\eta$.

The following sections give the reflection coefficients from such a material and a way to visualize them.

II. REFLECTION COEFFICIENTS

The geometry of the problem to be analyzed is very simple and shown in Figure 1. An incident electromagnetic wave is impinging from free space and faces a planar interface. On the other side of the boundary, there is a homogeneous half space of dielectric–magnetic medium with refractive index and impedance parameters n and η . After the collision with the boundary, part of the energy is refracted and penetrates into the medium, and the remaining part reflects away from the interface.

In general, the wave changes its polarization state in reflection. Only for two eigenpolarizations of the incident wave do the reflected and refracted waves remain with the same polarization as the incoming wave. These two are parallel (P) and perpendicular (S) polarizations, meaning that the linearly polarized electric field vector is in the plane of incidence (P) or perpendicular to it (S). The plane of incidence is spanned by the incident wave direction and the normal of the interface (the plane of paper in Figure 1).

The reflection coefficients for the two polarizations can be written in many equivalent forms [2, 3]; the following

*Electronic address: ari.sihvola@tkk.fi

†URL: <http://www.tkk.fi/~asihvola>

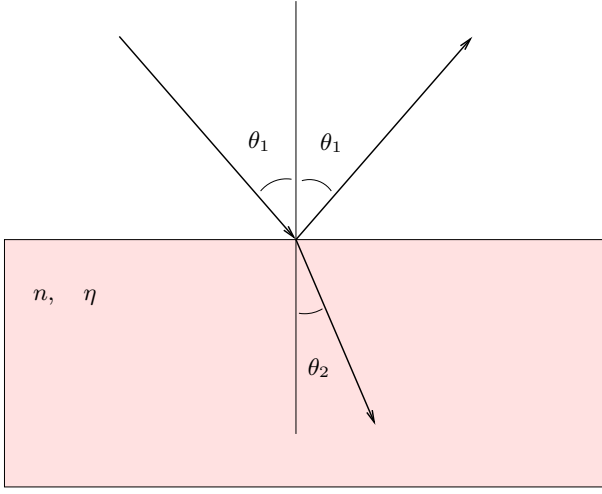


FIG. 1: Plane wave hitting a boundary between free space and a dielectric-magnetic material with refractive index n and impedance η .

electric field Fresnel coefficients are quite symmetric:

$$R_P = \frac{\eta \cos \theta_2 - \cos \theta_1}{\eta \cos \theta_2 + \cos \theta_1} \quad (2)$$

$$R_S = \frac{\eta \cos \theta_1 - \cos \theta_2}{\eta \cos \theta_1 + \cos \theta_2} \quad (3)$$

In using these formulas, the value for the refraction angle θ_2 is needed. It is determined by the Snell's law

$$\sin \theta_1 = n \sin \theta_2 \quad (4)$$

These expressions give the reflected electric field vector for unit incident field. The magnitudes of the reflection coefficients are always between zero and unity. Note, however, that the reflection coefficients can attain complex values even in the case of real values for n and η ; this happens for total internal reflection with the associated Goos-Hänchen phenomenon.

Of course, very interesting is the case when the reflection vanishes. It is easy to solve from (2)–(4) the incidence angle for which the reflection coefficient is zero. This is called the Brewster angle, and it is for the parallel polarization

$$\theta_{Br,P} = \arcsin \left(n \sqrt{\frac{1 - \eta^2}{n^2 - \eta^2}} \right) \quad (5)$$

For the perpendicular polarization the Brewster angle can be written as

$$\theta_{Br,S} = \arcsin \left(n \sqrt{\frac{\eta^2 - 1}{n^2 \eta^2 - 1}} \right) \quad (6)$$

Note that only for one polarization there exists a Brewster angle; the requirements are (see Figure 2)

- Parallel polarization: $n > 1$ and $\eta < 1$, or $n < 1$ and $\eta > 1$
- Perpendicular polarization: $n > 1$ and $\eta > 1$, or $n < 1$ and $\eta < 1$

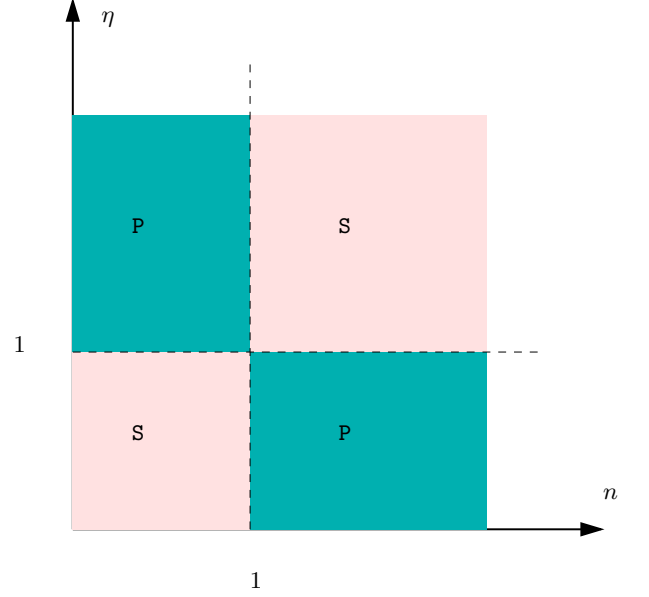


FIG. 2: Regions of the (n, η) -plane where the Brewster angle can be observed for parallel and perpendicular polarizations.

Note that the expression (5) is a generalization from the familiar Brewster-angle relation $\tan \theta_{Br,P} = n$ which is valid for non-magnetic media ($\eta = 1/n$), and naturally only exists for the parallel polarization. When magnetic response is allowed, the relation for the polarizing angle has one more degree of freedom. It can be written, of course, also in forms other than (5)–(6), see, for example [4].

An interesting observation is that the Brewster angle can attain any values between zero and 90° , as can be seen from Figure 3 in case of parallel polarization. Note that for ordinary dielectric materials where $n = 1/\eta$ the Brewster angle $\theta_{Br} = \arctan(n)$ is larger than 45° . For the parallel polarization, the impedance as function of the refractive index and the Brewster angle is

$$\eta = \frac{n \cos \theta_{Br}}{\sqrt{n^2 - \sin^2 \theta_{Br}}} \quad (7)$$

The simple law for the non-magnetic Brewster angle $\tan \theta_1 = n$, combined with the Snell's law $\sin \theta_1 = n \sin \theta_2$ yields $\cos \theta_1 = \sin \theta_2$. This means that the incidence and refracted angles are complementary angles ($\theta_1 + \theta_2 = 90^\circ$). Therefore (see Figure 1) the direction of the reflected wave is orthogonal to the refracted wave. In such a geometric constellation the dipoles induced in the medium by the refracted ray, which have a radiation null along their axis direction, do not cause reradiation

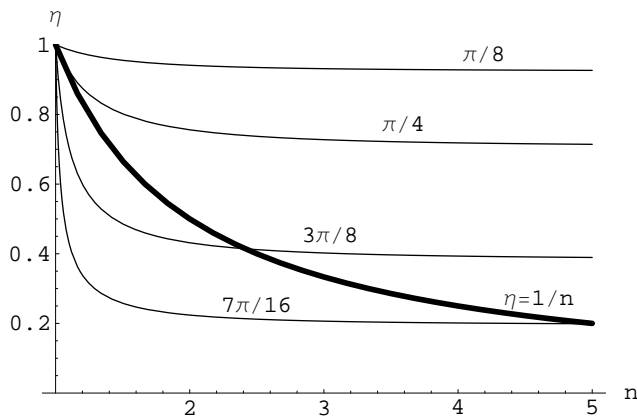


FIG. 3: Equi-Brewster-angle curves in the $(n-\eta)$ -plane for parallel polarization. Four curves are shown. The thick curve $\eta = 1/n$ divides the plane into a upper “paramagnetic part” where $\mu > 1$, and the lower “diamagnetic part” where $\mu < 1$.

into the direction of the reflected ray. Hence physical intuition agrees with the result of Brewster angle formula [5, 6], although the interpretation has been also criticized [7, 8].

But let us return to the more general case of the properties of the wave that reflects from a dielectric-magnetic interface.

III. GEOMETRIC INTERPRETATION

The square roots of differences of squares in the relations (5) and (6) for the two Brewster angles remind

of the Pythagorean theorem. And indeed, after some time of trigonometric play with these relations, beautiful geometric interpretations can be discovered from right triangles that are built from the three basic measures n , η , and $n\eta$. Further, an arrangement of these triangles in three dimensions reveals structures with which the Brewster angles can be grasped in a very visual sense.

This geometric construction is illustrated in Figure 4 for the relations expressing the Brewster angle for parallel polarization. From the magnitudes of n and η , a tetrahedron is uniquely determined. The faces of this geometrical object are four right triangles. The Brewster angle can be read from the bottom of the tetrahedron.

Figure 5 shows the same for the perpendicular polarization.

IV. CONCLUSION

Sir David Brewster performed his studies on the character of reflected light during the second decade of the 19th century. Therefore the concept of polarizing angle is nearly as old as the understanding of the transverse nature of light. The fascinating manner how the material properties affect the appearance of the Brewster angle is very interesting still today, both from experimental application point of view and also pedagogically when we are learning physics, optics, and electromagnetism. Hopefully the present article can give a helpful contribution to a modern understanding of the Brewster angle.

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- [11] It is perhaps important to emphasize here the explicit assumption of positiveness of the material parameters. In recent years very much research has been and is still being focused on materials with negative permittivity and permeability values, so-called negative-phase-velocity media, left-handed media, or metamaterials [9, 10]. Large research programs have been launched in the U.S. and in Europe which target on design and exploitation of metamaterials; see, for example, <http://www.darpa.mil/dso/thrust/matdev/metamat.htm> and <http://www.metamorphose-eu.org>

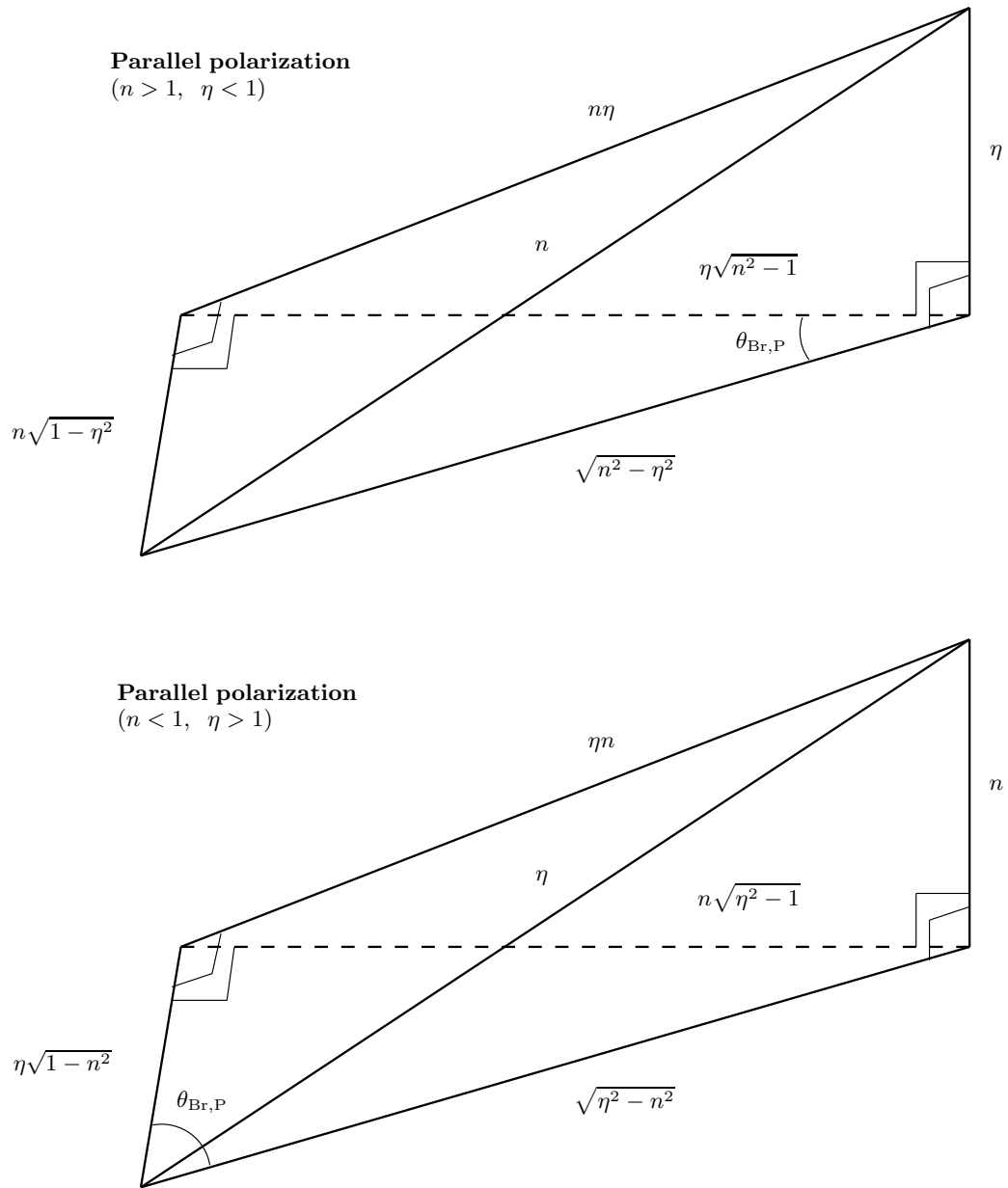


FIG. 4: A geometrical view of the Brewster angle determined by the primary material constants n and η . Parallel polarization, $n > 1, \eta < 1$ (upper figure); $n < 1, \eta > 1$ (lower figure). Note the four right-triangular faces of the tetrahedra.

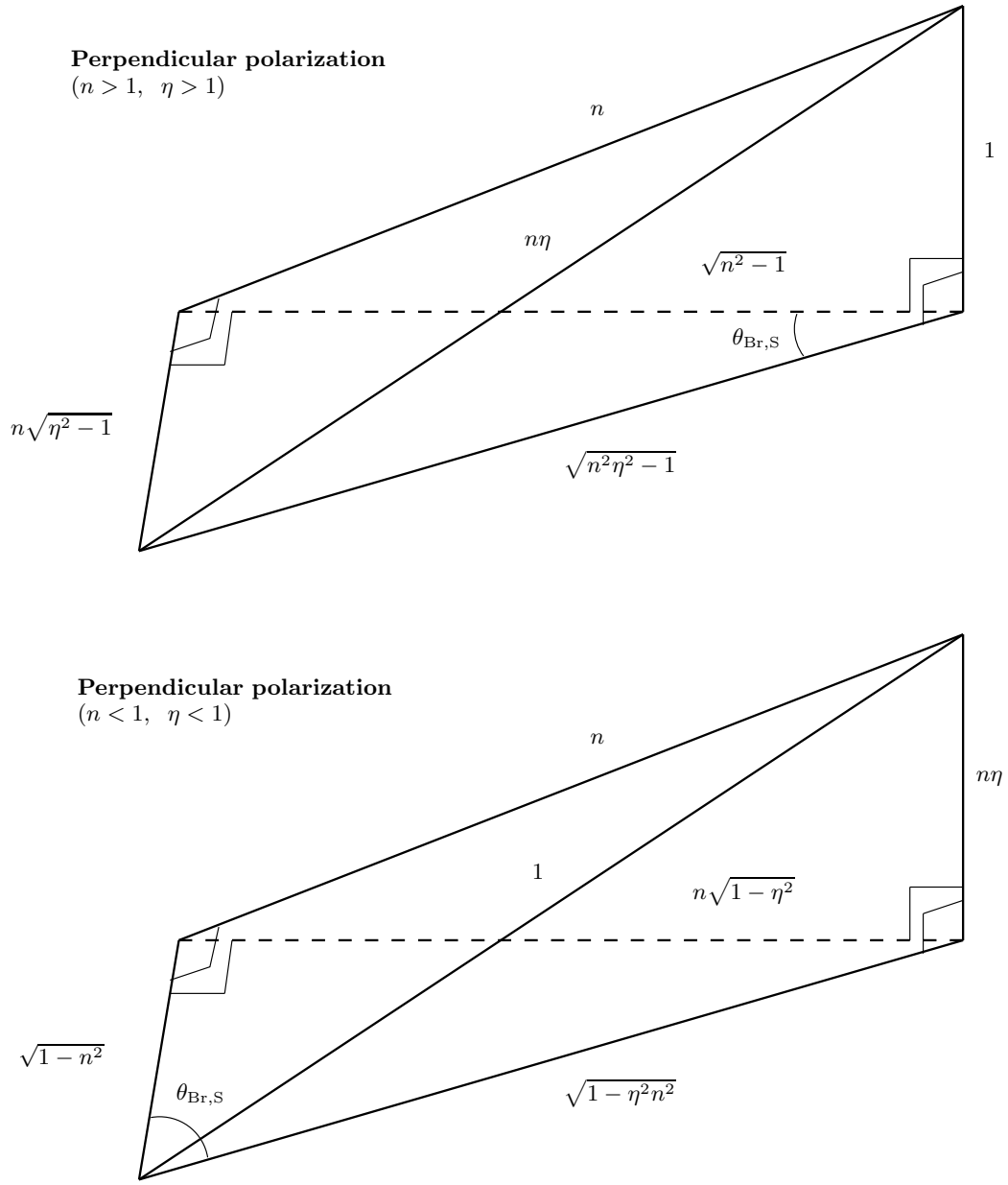


FIG. 5: The same as in Figure 4, for the perpendicular polarization. Upper figure: $n > 1, \eta > 1$; lower figure: $n < 1, \eta < 1$.