NORGES TEKNISK-
NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR FYSIKK

Kontakt under eksamen:
Jon Andreas Støvneng
Telefon: 73 59 36 63 / 45 45 55 33

KONTINUASJONSEKSAMEN TFY4160 BØLGEFYSIKK
Monday 11. august 2008 kl. 0900 - 1300

Hjelpemidler: C

- K. Rottmann: Matematisk formelsamling (alle språk).
- O. Øgrim og B. E. Lian: Størrelser og enheter i fysikk og teknikk, eller C. Angell og B. E. Lian: Fysiske størrelser og enheter.
- Typegodkjent kalkulator, med tomt minne, i henhold til liste utarbeidet av NTNU. (HP30S eller lignende.)

Page 2 - 5: The questions.
Page 6 - 14: Appendix, Formulas.

The exam consists of 6 questions. Vectors are given in bold letters. Unit vectors are given with a hat above the symbol.

The marks are ready no later then September 1.

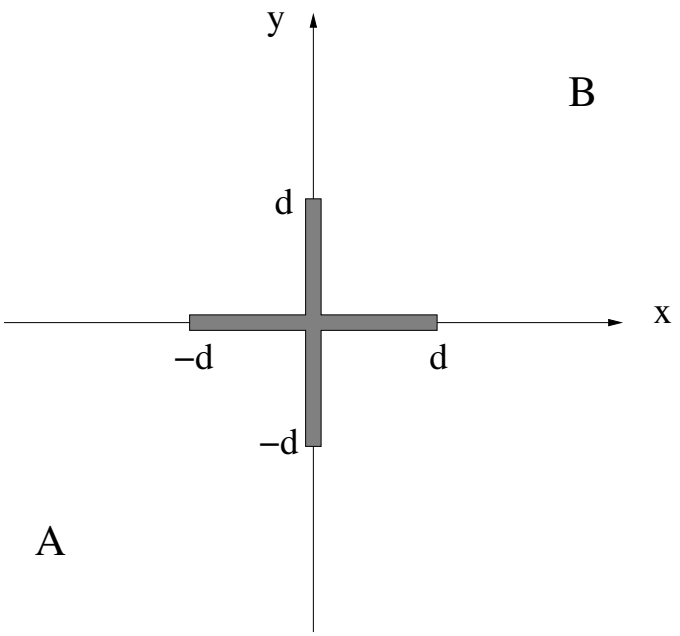
QUESTION 1 [1a and 1b count 10% each]

a) A guitar string with length $L = 70$ cm is fixed in both ends (ie nodes there). The fundamental (ie the lowest resonance frequency) and the harmonics are generated by standing transversal waves on the string. The string is supposed to be tuned such that the fundamental is a G with frequency $f = 196$ Hz. The string has circular cross section with radius $R = 0.25$ mm and is made out of steel, with mass density $\rho = 7800$ kg/m^3.

• Find the wave length λ and the wave velocity v of the 1. harmonic (ie second lowest resonance frequency). What is now the stretching force S on the string?

• You and your friend Ole are on separate wagons rolling away from each other, with speed (relative to the ground) 10 m/s (you) and 25 m/s (Ole). Ole plays the G-string of his guitar. What frequency will the fundamental have in your ears? (The sound from the guitar propagates in air with speed 340 m/s.)

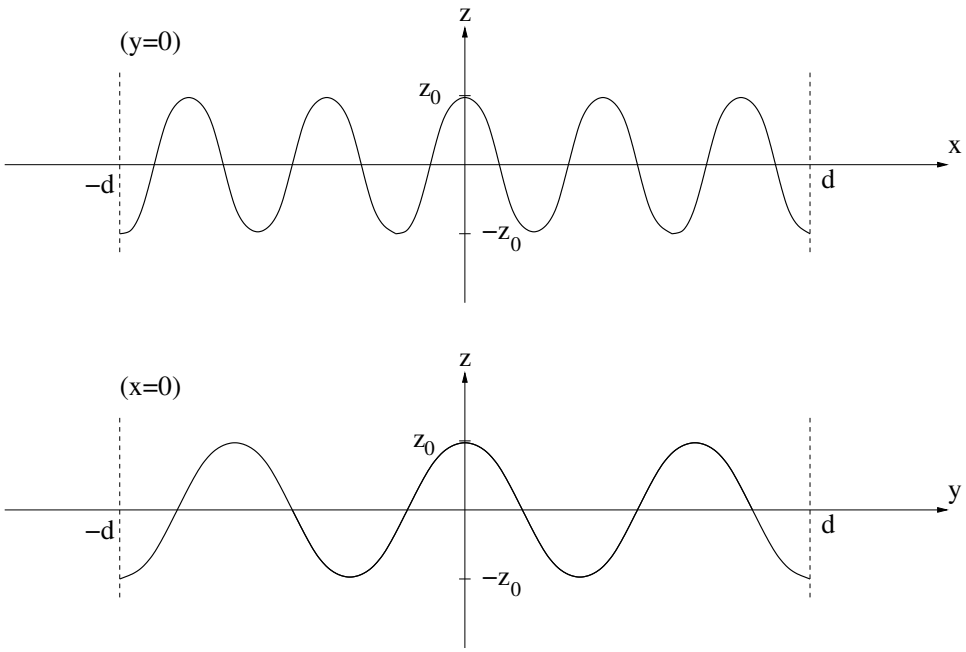
b) A thin membrane (eg paper) is located in the xy plane. A laser optical instrument can measure transversal displacements on the membrane, for $-d < x < d$ when $y = 0$ and for $-d < y < d$ when $x = 0$:



The membrane is large, both in the x and the y direction, when compared with d. A plane harmonic transversal wave (z = displacement, $\boldsymbol{r} = x\,\hat{x} + y\,\hat{y}$),

$$z(\boldsymbol{r}, t) = z_0 \cos\left(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t\right),$$

propagates in the membrane, with direction from region A towards region B (ie to the right and upwards in the figure). At time $t = 0$ the instrument displays the following displacement:



• Suppose $d = 40$ cm. Find the wave length λ.
• Find the propagation direction of the wave. Denote this direction by the angle with respect to the positive x axis.
• With another instrument, the frequency of the wave has been measured to be $\nu = 108$ Hz. What is the speed of the wave, $\boldsymbol{v} = v_x\,\hat{x} + v_y\,\hat{y}$? (Find both v, v_x and v_y.)

QUESTION 2 [counts 15%]

A two-atomic molecule (eg HCl) may, classically, be regarded as two masses, M and m, connected with a (massless) spring with spring constant k.
• Show, starting from Newton's 2. law, that the atoms in such a molecule may oscillate around their equilibrium positions with angular frequency

$$\omega = \sqrt{\frac{k}{\mu}}$$

where $\mu = mM/(m+M)$ is the reduced mass of the molecule.
Hint: Assume that the mass center of the molecule is at rest, and that the displacements of the atoms relative to their equilibrium positions X_0 and x_0 may be written as

$$\Xi(t) = X(t) - X_0 = A\cos\omega t$$

and

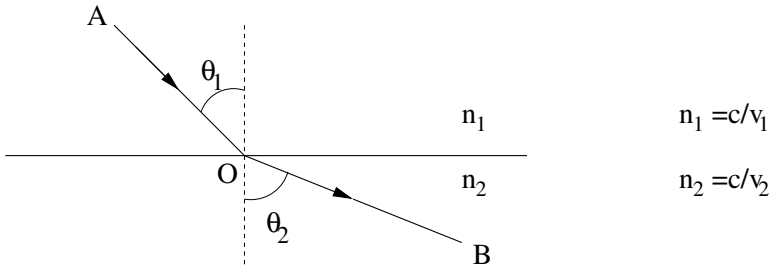
$$\xi(t) = x(t) - x_0 = B\cos\omega t$$

for M and m, respectively.

• Find the amplitude ratio $|A/B|$? (Actually, it may be natural to determine $|A/B|$ before you determine ω.)

QUESTION 3 [counts 10%]

Fermat's principle states that light chooses the path between two points A and B corresponding to the shortest time. Use this principle to derive Snell's law of refraction, $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Start from the figure below, where we consider light propagating from A to B via a position O in the boundary between the two media.



QUESTION 4 [counts 20%]

• Show that an electromagnetic wave in vacuum, with

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

and

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

propagates in the direction given by $\boldsymbol{k}$.

• Use Maxwell's equations to show that such an electromagnetic wave is transverse, ie, that $\boldsymbol{k} \perp \boldsymbol{E}$ and $\boldsymbol{k} \perp \boldsymbol{B}$.

• Also show that $\boldsymbol{E} \perp \boldsymbol{B}$.

QUESTION 5 [counts 25%]

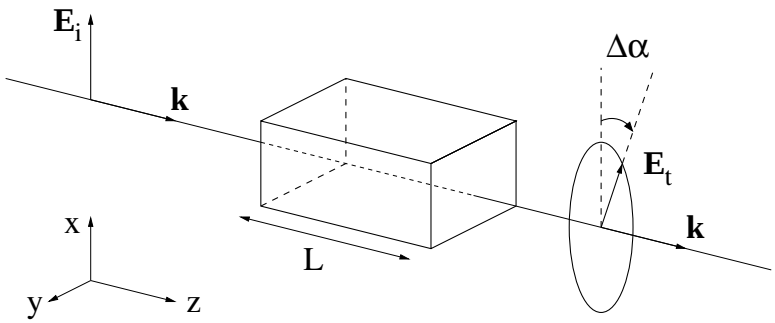
Two linearly polarized electromagnetic waves may be superimposed to form circularly polarized waves:

$$\boldsymbol{E}_{\pm}(z,t) = E_0\hat{x}\sin(kz - \omega t) \pm E_0\hat{y}\cos(kz - \omega t)$$

• What is the propagation direction of the waves $\boldsymbol{E}_+$ and $\boldsymbol{E}_-$?

• Show that a linearly polarized wave polarized in (eg) the x direction may be regarded as a superposition of a "left-rotating" ($\boldsymbol{E}_+$) and a "right-rotating" ($\boldsymbol{E}_-$) circularly polarized wave.

A certain type of molecules are so-called optically active. In a gas consisting of such molecules, the refractive indices n_+ and n_- for left- and right-rotating circularly polarized electromagnetic waves are different.



• A container of length L is filled with a gas of optically active molecules, with $\Delta n = n_+ - n_- > 0$. A left- and a right-rotating circularly polarized electromagnetic wave spends the time t_+ and t_-, respectively, propagating through the container. Determine the time difference $\Delta t = t_+ - t_-$.
• A linearly polarized electromagnetic wave $\boldsymbol{E}_i$ is polarized in the x direction and is sent in towards the container described above (wave vector $\boldsymbol{k} = 2\pi\hat{z}/\lambda$). Show that the transmitted wave $\boldsymbol{E}_t$ is linearly polarized, but with the polarization no longer along the x axis. Find an expression for the angle $\Delta\alpha$ between the polarization direction of the incident wave and the transmitted wave. Determine numerical value for $\Delta\alpha$ when $L = 0.50$ m, $\lambda = 628$ nm and $\Delta n = 4.0 \cdot 10^{-9}$.

Given information:

$$\begin{aligned} \sin(a-b) + \sin(a+b) &= 2\sin a \cos b \\ \cos(a-b) - \cos(a+b) &= 2\sin a \sin b \end{aligned}$$

QUESTION 6 [counts 10%]

• Use the Lorentz transformations to derive Einstein's formula for addition of velocities,

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}/c^2}$$

Here, v_{ij} is the velocity of inertial system i as measured in system j $(i, j = A, B, C)$, and we assume all relative velocities v_{ij} along the same axis.
• Assume $v_{AB} = c/2$ and $v_{BC} = c/3$, both along the positive x axis. Then, what is the error, percentagewise, in v_{AC} if we don't use relativity and simply use $v_{AC} = v_{AB} + v_{BC}$?

Formulas

Bold symbols denote vectors. Symbols with hat above are unit vectors. Validity of formulas and meaning of symbols are assumed to be known.

- Harmonic plane wave:

$$\xi(x,t) = \xi_0 \sin(kx - \omega t + \phi)$$
$$\xi(\boldsymbol{r},t) = \xi_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \phi)$$

- Wave equation:

$$\frac{\partial^2 \xi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi(x,t)}{\partial t^2}$$
$$\nabla^2 \xi(\boldsymbol{r},t)\left(\equiv \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right) = \frac{1}{v^2}\frac{\partial^2 \xi(\boldsymbol{r},t)}{\partial t^2}$$

- Phase velocity:

$$v = \frac{\omega}{k}$$

- Group velocity:

$$v_g = \frac{d\omega}{dk}$$

- General, for nondispersive undamped waves:

$$v = \sqrt{\frac{\text{elastic modulus}}{\text{massdensity}}}$$

- General, for linear response in elastic media:

$$\text{mechanical stress} = \text{elastic modulus} \times \text{relative strain}$$

- For transverse waves on string:

$$v = \sqrt{\frac{S}{\mu}}$$

- For longitudinal waves in fluids:

$$v = \sqrt{\frac{B}{\rho}}$$

- For longitudinal waves in solids:

$$v = \sqrt{\frac{Y}{\rho}}$$

- Average of harmonically varying $A(x,t)$, averaged over wave length λ:

$$\overline{A} = \frac{\int_0^\lambda A(x,t)\,dx}{\int_0^\lambda dx} = \frac{1}{\lambda}\int_0^\lambda A(x,t)\,dx$$

  Average of harmonically varying $A(x,t)$, averaged over period T:

$$\langle A\rangle = \frac{\int_0^T A(x,t)\,dt}{\int_0^T dt} = \frac{1}{T}\int_0^T A(x,t)\,dt$$

- Average energy pr unit length for harmonic wave on string:

$$\overline{\varepsilon} = \frac{1}{2}\mu\omega^2\xi_0^2$$

- Average energy pr unit volume for harmonic plane wave:

$$\overline{\varepsilon} = \frac{1}{2}\rho\omega^2\xi_0^2$$

- Average power transported with harmonic wave on string:

$$\overline{P} = v\overline{\varepsilon} = \frac{1}{2}v\mu\omega^2\xi_0^2$$

- Average intensity in harmonic plane wave:

$$I = v\overline{\varepsilon} = \frac{1}{2}v\rho\omega^2\xi_0^2$$

- Average momentum density for harmonic wave:

$$\overline{\pi} = \frac{\overline{\varepsilon}}{v}$$

- Ideal gas:

$$pV = Nk_BT$$

- Heat capacity at constant pressure (Q = heat):

$$C_p = \left(\frac{dQ}{dT}\right)_p$$

- Heat capacity at constant volume (Q = heat):

$$C_V = \left(\frac{dQ}{dT}\right)_V$$

- Adiabatic conditions (i.e. no heat exchange):

$$pV^\gamma = \text{constant}$$

- The adiabatic constant:

$$\gamma = \frac{C_p}{C_V}$$

  Gas with 1-atomic molecules: $\gamma = 5/3$. Gas with 2-atomic molecules: $\gamma = 7/5$.

- Bulk modulus for ideal gas at adiabatic conditions:

$$B = \gamma p$$

- Sound velocity in gas (m = molecular mass):

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma k_B T}{m}}$$

- Sound pressure:

$$\Delta p = -B\frac{\partial \xi}{\partial x}$$

- Sound level:

$$\beta(\text{dB}) = 10 \log \frac{I}{I_0}$$

  with $I_0 = 10^{-12}$ W/m^2

- Doppler effect:

$$\nu_\text{O} = \frac{1 - v_\text{O}/v}{1 - v_\text{S}/v}\,\nu_\text{S}$$

- For shock waves:

$$\sin\alpha = \frac{v}{v_\text{S}}$$

- Transverse wave on string with mass density μ_1 for $x < 0$ and μ_2 for $x > 0$, incident wave propagating in positive x direction:

  Amplitude for reflected wave:

  $$y_{r0} = \frac{\sqrt{\mu_2} - \sqrt{\mu_1}}{\sqrt{\mu_2} + \sqrt{\mu_1}}\, y_{i0}$$

  Amplitude for transmitted wave:

  $$y_{t0} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_2} + \sqrt{\mu_1}}\, y_{i0}$$

  Reflection coefficient:

  $$R = \frac{\overline{P}_r}{\overline{P}_i}$$

  Transmission coefficient:

  $$T = \frac{\overline{P}_t}{\overline{P}_i}$$

- Plane sound wave with normal incidence on boundary in $x = 0$ between two media with elastic moduli and mass densities E_1, ρ_1 (for $x < 0$) and E_2, ρ_2 (for $x > 0$), respectively. Incident wave propagating in positive x direction:

  Amplitude for reflected wave:

  $$\xi_{r0} = \frac{\sqrt{\rho_2 E_2} - \sqrt{\rho_1 E_1}}{\sqrt{\rho_2 E_2} + \sqrt{\rho_1 E_1}}\, \xi_{i0}$$

  Amplitude for transmitted wave:

  $$\xi_{t0} = \frac{2\sqrt{\rho_1 E_1}}{\sqrt{\rho_2 E_2} + \sqrt{\rho_1 E_1}}\, \xi_{i0}$$

  Reflection coefficient:

  $$R = \frac{\overline{P}_r}{\overline{P}_i}$$

  Transmission coefficient:

  $$T = \frac{\overline{P}_t}{\overline{P}_i}$$

- Maxwell's equations on integral form:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = q/\varepsilon_0$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{A} = 0$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \int \boldsymbol{B} \cdot d\boldsymbol{A}$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d}{dt} \int \boldsymbol{E} \cdot d\boldsymbol{A}$$

- Maxwell's equations on differential form:

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

- Lorentz force:

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$$

- Wave equation for $\boldsymbol{E}$ and $\boldsymbol{B}$ in vacuum:

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{B} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

$$c = 1/\sqrt{\varepsilon_0 \mu_0}$$

- Energy density in electromagnetic field:

$$u = u_E + u_B = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

- Intensity in electromagnetic wave:

$$I = c\varepsilon_0 \overline{E^2} = c\varepsilon_0 \langle E^2 \rangle$$

- Poynting's vector:

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$$

- Momentum in electromagnetic wave:

$$\boldsymbol{\pi} = \mu_0 \varepsilon_0 \boldsymbol{S}$$

- Electric dipole moment:

$$\boldsymbol{p} = q\boldsymbol{d}$$

- Magnetic dipole moment:

$$\boldsymbol{m} = I\boldsymbol{A}$$

- Average radiated effect from oscillating electric dipole $p_0 \cos(\omega t)$:

$$\langle P \rangle = \frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3}$$

- Average radiated effect from oscillating magnetic dipole $m_0 \cos(\omega t)$:

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

- Malus' law:

$$I(\theta) = I_0 \cos^2\theta$$

- Linear media:

$$\begin{aligned}
\boldsymbol{P} &= \varepsilon_0 \chi_e \boldsymbol{E} \\
\boldsymbol{D} &= \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0(1+\chi_e)\boldsymbol{E} = \varepsilon_0\varepsilon_r\boldsymbol{E} = \varepsilon\boldsymbol{E} \\
\boldsymbol{M} &= \chi_m \boldsymbol{H} \\
\boldsymbol{B} &= \mu_0 \boldsymbol{H} + \boldsymbol{M} = \mu_0(1+\chi_m)\boldsymbol{H} = \mu_0\mu_r\boldsymbol{H} = \mu\boldsymbol{H}
\end{aligned}$$

- Maxwell's equations etc:

$$\begin{aligned}
\oint \boldsymbol{D} \cdot d\boldsymbol{A} &= q_{\text{free}} \\
\oint \boldsymbol{B} \cdot d\boldsymbol{A} &= 0 \\
\oint \boldsymbol{E} \cdot d\boldsymbol{l} &= -\frac{d}{dt}\int \boldsymbol{B} \cdot d\boldsymbol{A} \\
\oint \boldsymbol{H} \cdot d\boldsymbol{l} &= I_{\text{free}} + \frac{d}{dt}\int \boldsymbol{D} \cdot d\boldsymbol{A} \\
\nabla \cdot \boldsymbol{D} &= \rho_{\text{free}} \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \times \boldsymbol{H} &= \boldsymbol{j}_{\text{free}} + \frac{\partial \boldsymbol{D}}{\partial t}
\end{aligned}$$

- Energy density, Poynting's vector:

$$\begin{aligned}
u &= \frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}B^2 \\
\boldsymbol{S} &= \frac{1}{\mu}\boldsymbol{E} \times \boldsymbol{B}
\end{aligned}$$

- For electromagnetic waves in media ($q_{\text{free}} = I_{\text{free}} = 0$):

$$\begin{aligned} \nabla^2 \boldsymbol{E} &= \frac{1}{v^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \\ \nabla^2 \boldsymbol{B} &= \frac{1}{v^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} \\ v &= \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{c}{n} \end{aligned}$$

- Boundary conditions ($q_{\text{free}} = I_{\text{free}} = 0$ in the boundary):

$$\begin{aligned} \Delta D_\perp &= 0 \\ \Delta E_\parallel &= 0 \\ \Delta B_\perp &= 0 \\ \Delta H_\parallel &= 0 \end{aligned}$$

- Reflection and refraction:

$$\theta_r = \theta_i$$
$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

- Young's experiment with two narrow slits:

$$I(\theta) = 4I_0 \cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$$

- Diffraction grating with N narrow slits:

$$I(\theta) = I_0 \, \frac{\sin^2\left(\frac{N\pi d}{\lambda}\sin\theta\right)}{\sin^2\left(\frac{\pi d}{\lambda}\sin\theta\right)}$$

- Diffraction from one slit:

$$I(\theta) = I(0) \, \frac{\sin^2\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\left(\frac{\pi a}{\lambda}\sin\theta\right)^2}$$

- Diffraction from N slits of width a:

$$I(\theta) = I_0 \, \frac{\sin^2\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\left(\frac{\pi a}{\lambda}\sin\theta\right)^2} \; \frac{\sin^2\left(\frac{N\pi d}{\lambda}\sin\theta\right)}{\sin^2\left(\frac{\pi d}{\lambda}\sin\theta\right)}$$

- Lorentz factor:

$$\gamma = \left(1 - v^2/c^2\right)^{-1/2}$$

- Lorentz transformations ($\overline{S}$ has velocity $\boldsymbol{v} = v\hat{x}$ relative to S):

$$\begin{aligned} \overline{x} &= \gamma\left(x - vt\right) \\ \overline{y} &= y \\ \overline{z} &= z \\ \overline{t} &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}$$

$$\begin{aligned} x &= \gamma\left(\overline{x} + v\overline{t}\right) \\ y &= \overline{y} \\ z &= \overline{z} \\ t &= \gamma\left(\overline{t} + \frac{v}{c^2}\overline{x}\right) \end{aligned}$$

- Time dilation:

$$\Delta t = \gamma \Delta \overline{t}$$

- Length contraction:

$$\Delta \overline{x} = \gamma \Delta x$$

- Velocity in S ($\boldsymbol{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$):

$$u_x = dx/dt$$

$$u_y = dy/dt$$

$$u_z = dz/dt$$

  Velocity in $\overline{S}$ ($\overline{\boldsymbol{u}} = \overline{u}_x\hat{x} + \overline{u}_y\hat{y} + \overline{u}_z\hat{z}$):

$$\overline{u}_x = d\overline{x}/d\overline{t}$$

$$\overline{u}_y = d\overline{y}/d\overline{t}$$

$$\overline{u}_z = d\overline{z}/d\overline{t}$$

- Addition of velocities (all velocities in the same direction):

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}/c^2}$$

- Doppler effect for electromagnetic waves:

$$\overline{\nu} = \nu\left(\frac{c - v}{c + v}\right)^{1/2}$$

- Relativistic momentum:

$$\boldsymbol{p} = \gamma m \boldsymbol{v}$$

- Newton's 2. law:

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

- Energy:

$$\begin{aligned} E &= \gamma m c^2 \\ E_0 &= m c^2 \\ E_k &= E - E_0 \\ E^2 &= (pc)^2 + \left(mc^2\right)^2 \end{aligned}$$

- Elastic process: E, $\boldsymbol{p}$, E_k and m conserved.
- Inelastic process: E and $\boldsymbol{p}$ conserved.