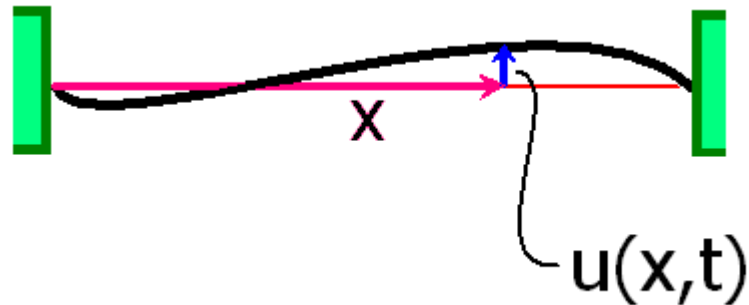


Free vibration of a beam as
compared to a taut cable



A “cable” in this context implies no stiffness; the restoring force comes from the fact that the cable is in tension (assumed constant) T .

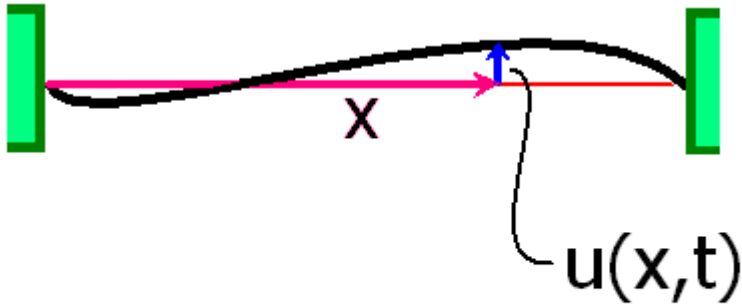


A “beam” on the other hand possesses stiffness, EI , but has no tension.

Free vibration of a beam as compared to a taut cable

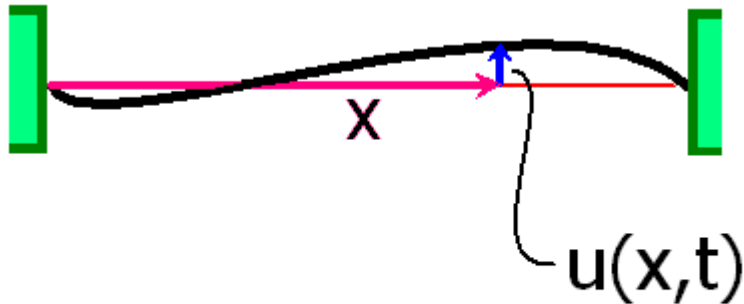
Many structural members in fact possess a combination of tension (compression) *and* stiffness.

Free vibration of a beam as compared to a taut cable



Recall that the vibrating cable or string entailed a second "independent variable" besides time t . This is denoted as " x ".

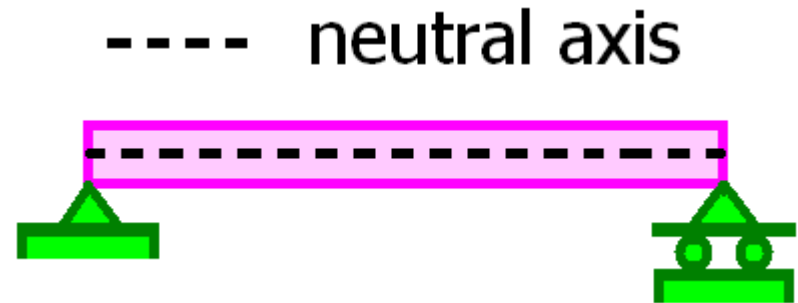
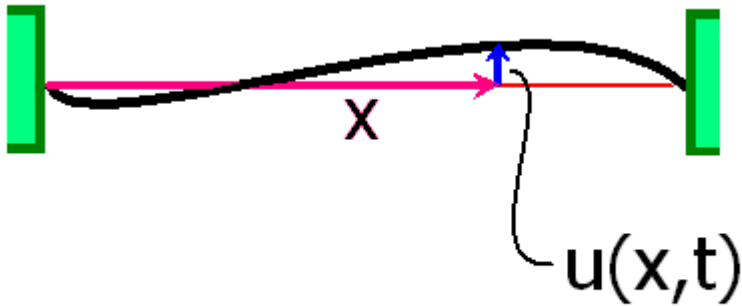
Free vibration of a beam as compared to a taut cable



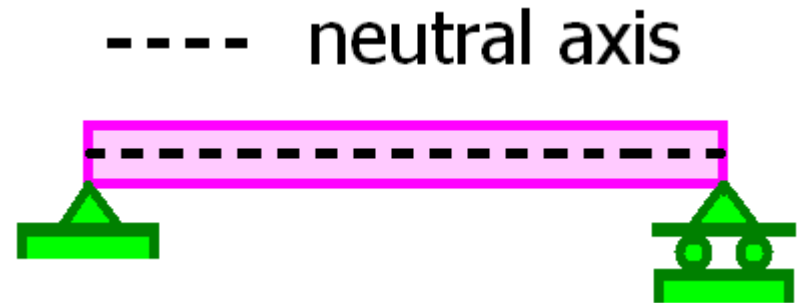
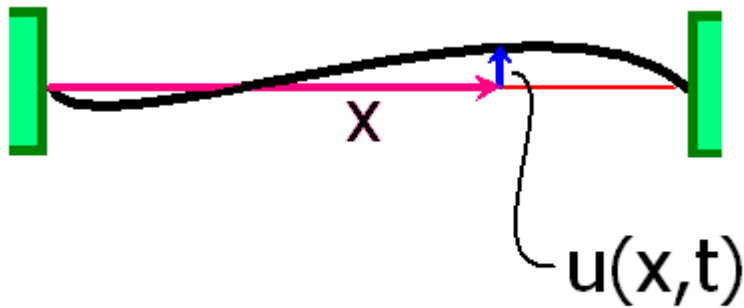
Recall that the vibrating cable or string entailed a second "independent variable" besides time t . This is denoted as " x ".

So the deflection is $u(x,t)$.

Free vibration of a beam as compared to a taut cable

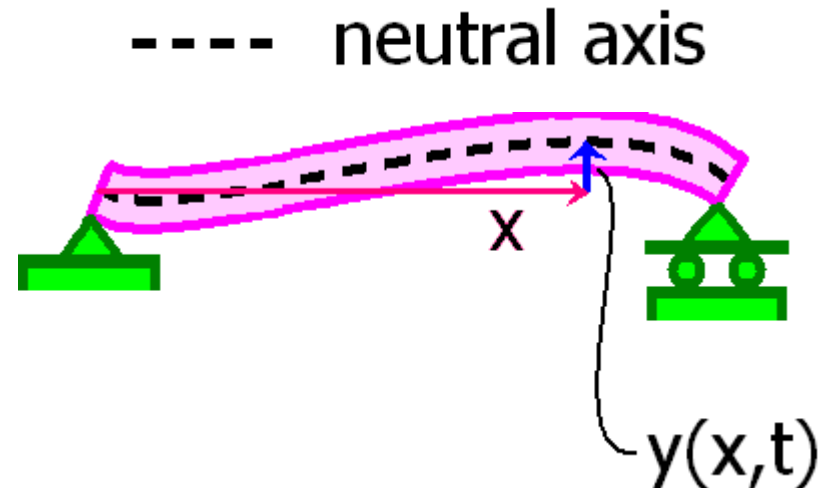
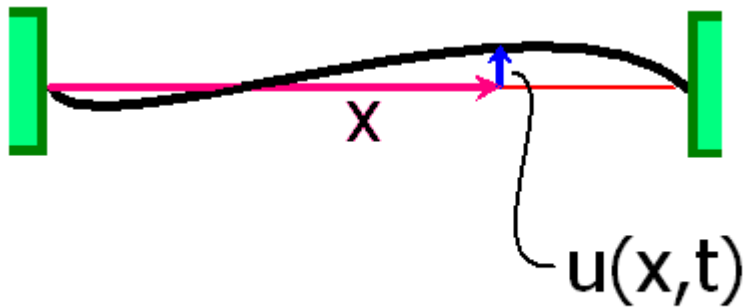


Free vibration of a beam as compared to a taut cable



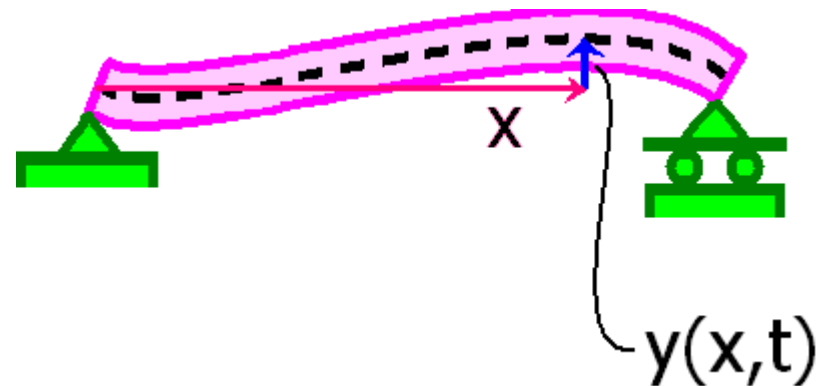
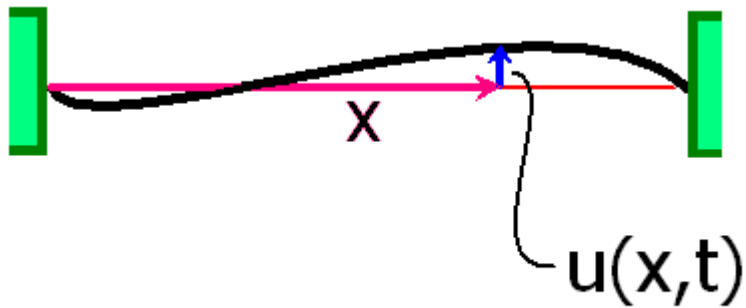
With the beam, we keep track of the deflection of the neutral axis.

Free vibration of a beam as compared to a taut cable

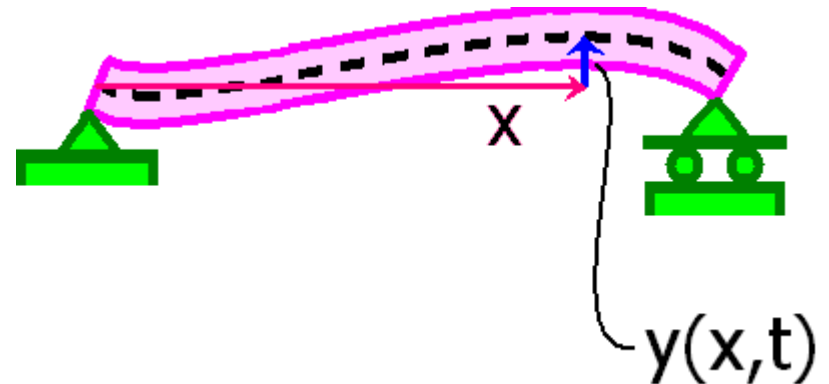
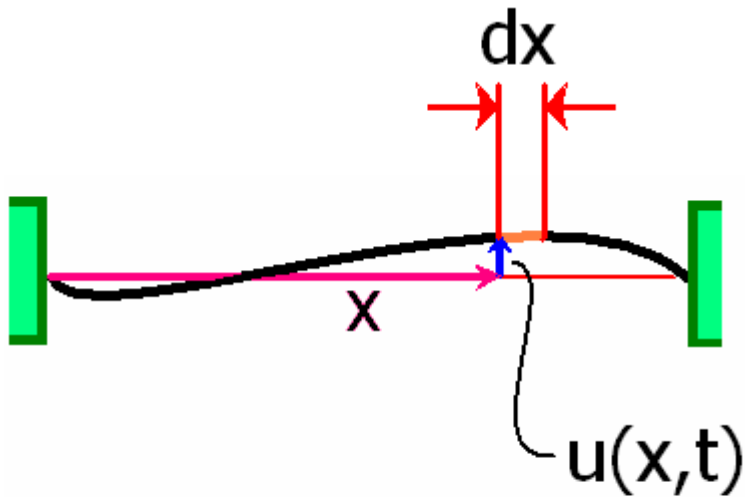


In this case we denote the deflection of the neutral axis by $y(x,t)$.

Comparing the governing partial differential equations

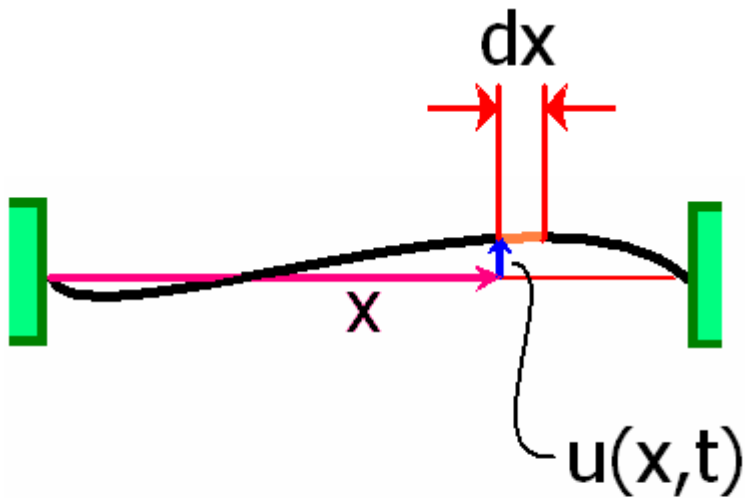


Comparing the governing partial differential equations

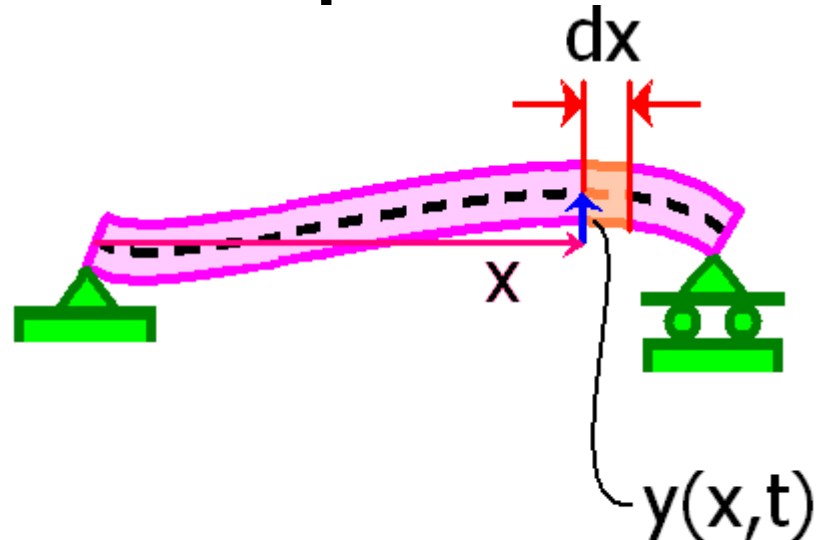


Consider a differential element of the cable, dx .

Comparing the governing partial differential equations

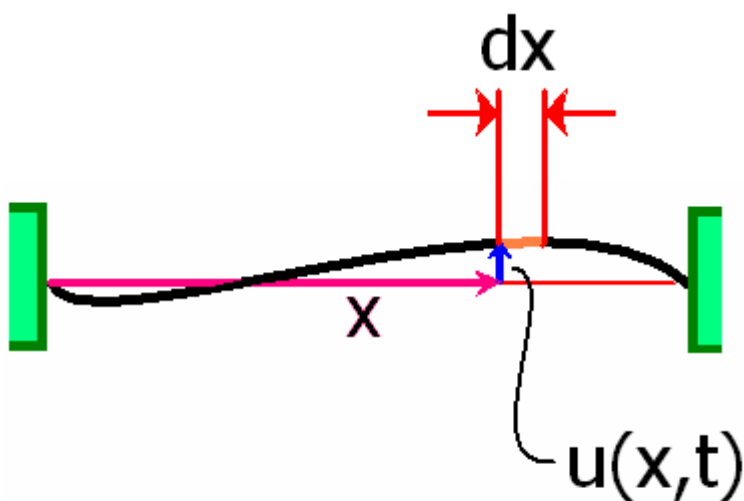


Consider a differential element of the cable, dx .

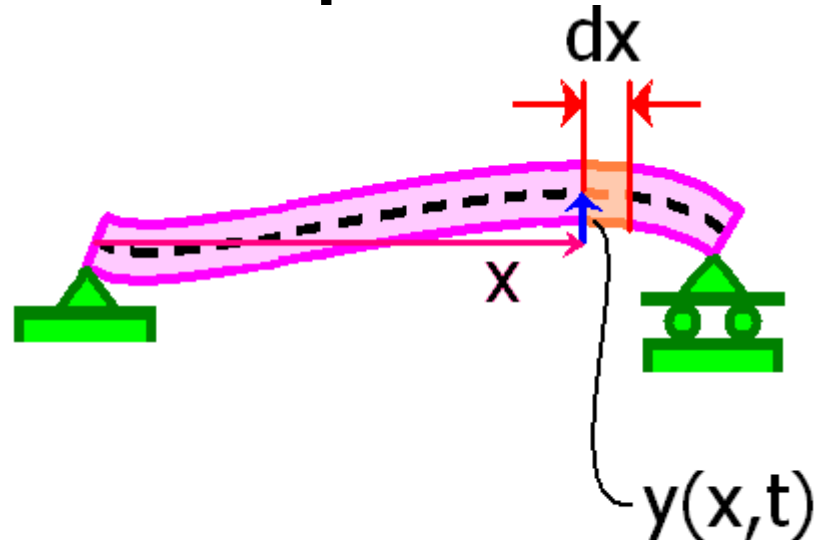


Also consider a differential element dx of the beam.

Comparing the governing partial differential equations



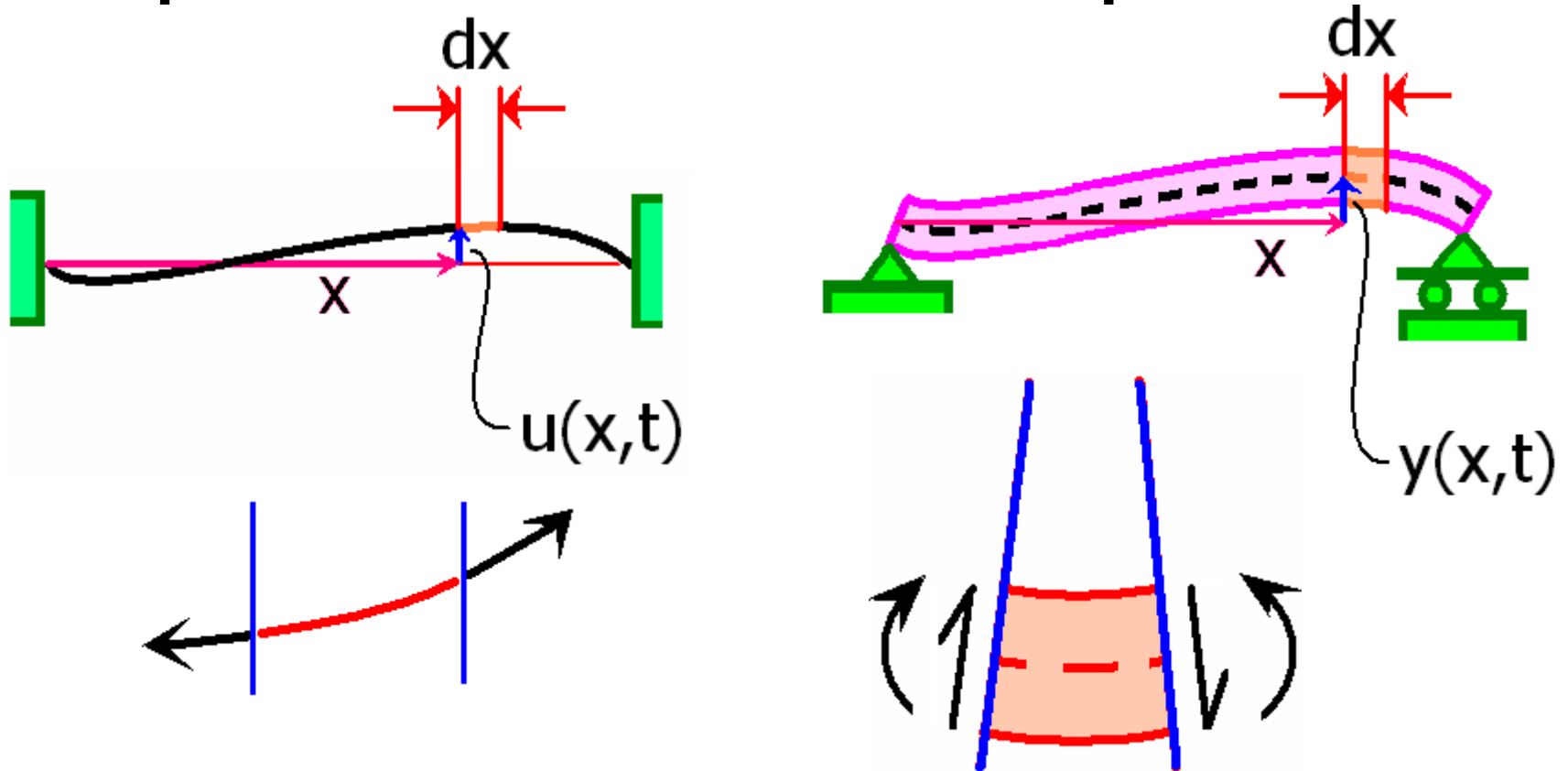
Consider a differential element of the cable, dx .



Also consider a differential element dx of the beam.

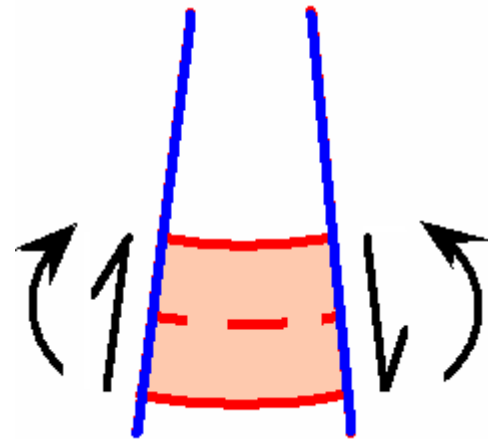
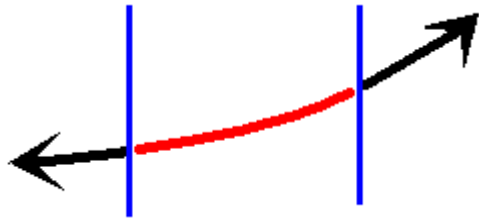
In both cases neglecting gravity, we consider a free body diagram of the two elements.

Comparing the governing partial differential equations

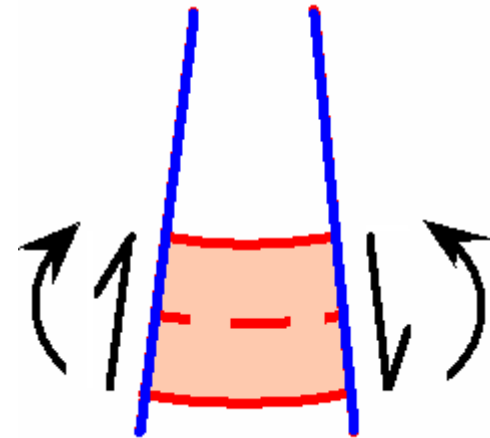
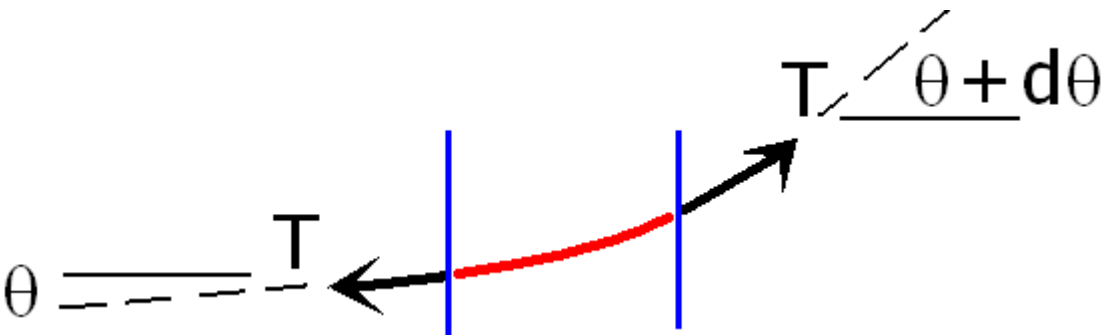


In both cases neglecting gravity, we consider a free body diagram of the two elements.

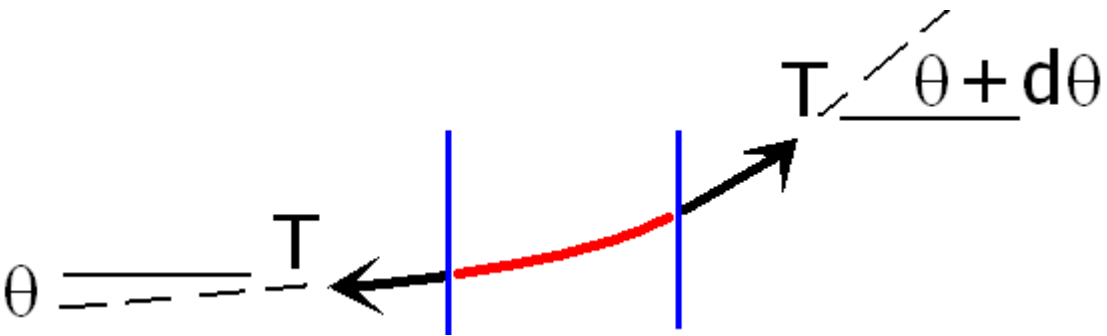
Free body diagrams



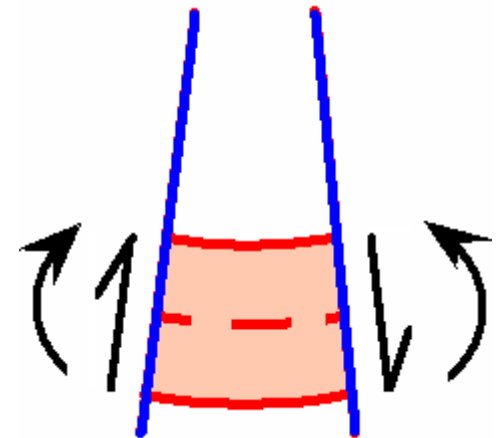
Free body diagrams



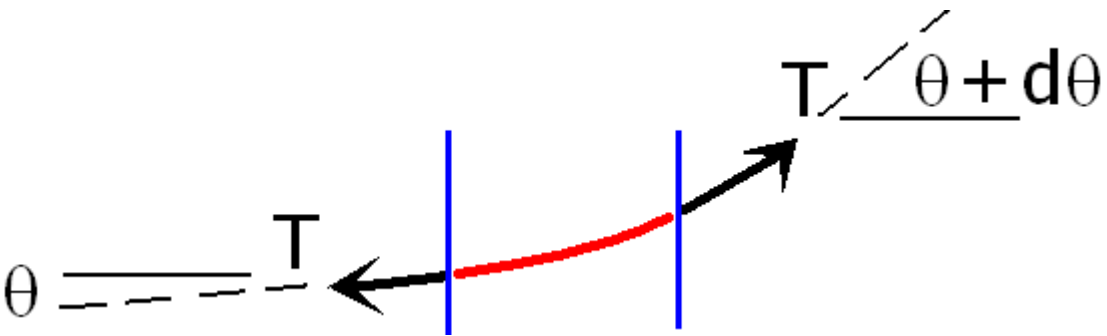
Free body diagrams



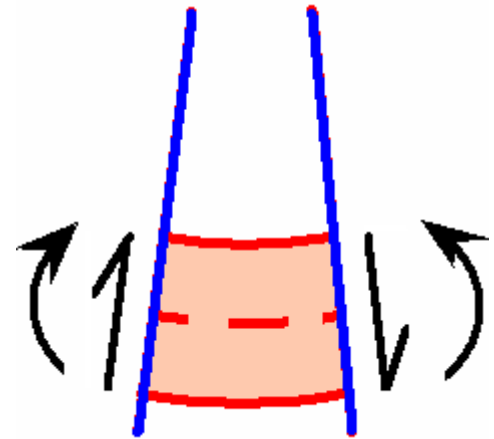
$$\begin{aligned}\Sigma F_{\text{vert.}} &= (m/L)dx \frac{d^2 u}{dt^2} \\ &= -T \sin(\theta) + T \sin(\theta + d\theta)\end{aligned}$$



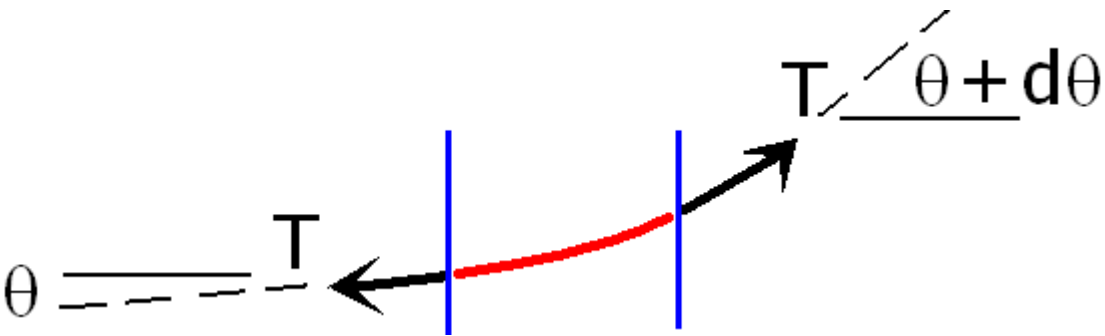
Free body diagrams



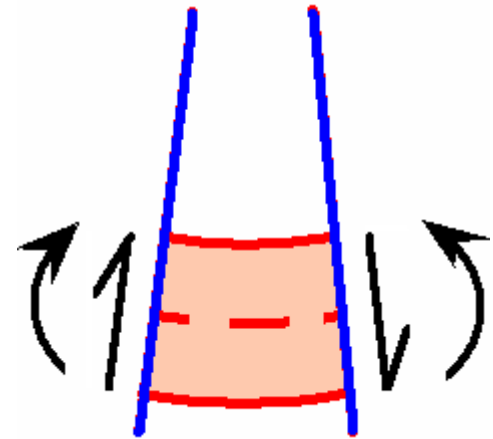
$$\begin{aligned}\Sigma F_{\text{vert.}} &= (m/L) dx \frac{d^2 u}{dt^2} \\ &= -T \sin(\theta) + T \sin(\theta + d\theta)\end{aligned}$$



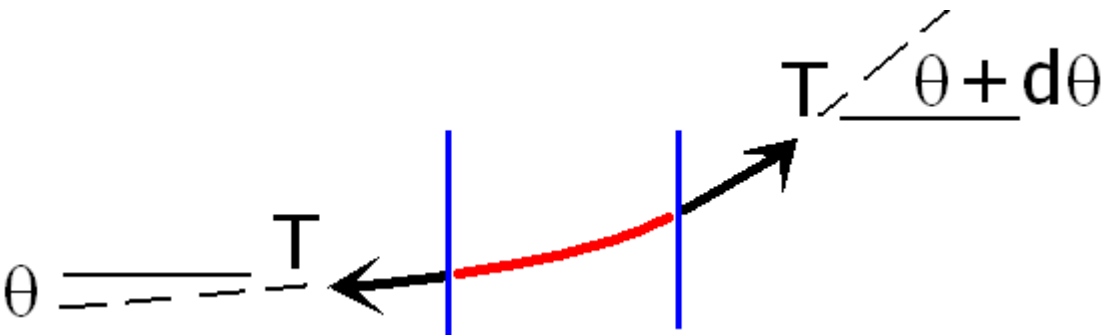
Linearizing about small θ



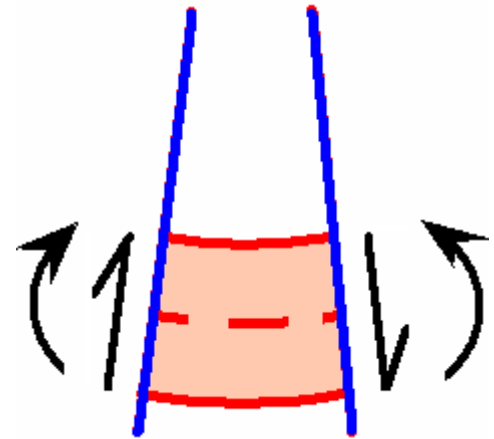
$$\begin{aligned}\Sigma F_{\text{vert.}} &= (m/L) \partial_x \frac{\partial^2 u}{\partial t^2} \\ &= -T \sin(\theta) + T \sin(\theta + \partial\theta) \\ &\approx T \partial\theta\end{aligned}$$



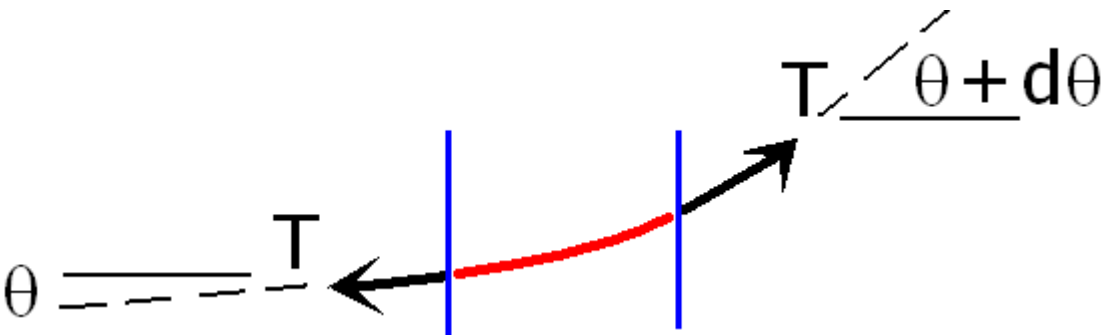
Linearizing about small θ



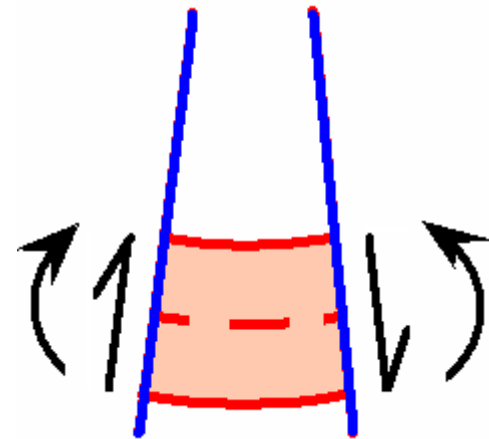
$$\begin{aligned}\Sigma F_{\text{vert.}} &= (m/L) \partial_x \frac{\partial^2 u}{\partial t^2} \\ &= -T \sin(\theta) + T \sin(\theta + \partial\theta) \\ &\approx T \partial\theta \approx T \partial_x \frac{\partial u}{\partial x}\end{aligned}$$



Finally, for the cable



$$\frac{\partial^2 u}{\partial t^2} = \frac{TL}{m} \frac{\partial^2 u}{\partial x^2}$$

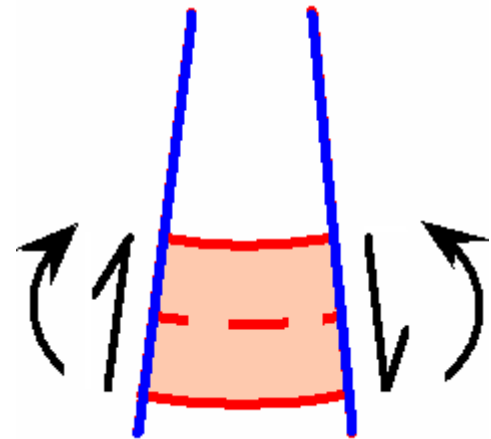


Finally, for the cable

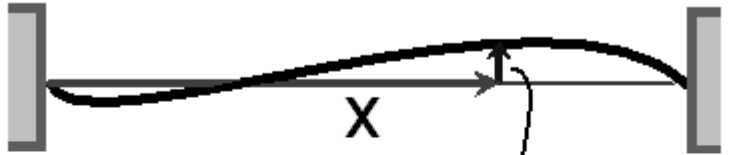


$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

"wave equation" w/
boundary conditions:
 $u(0,t)=u(L,t)=0$

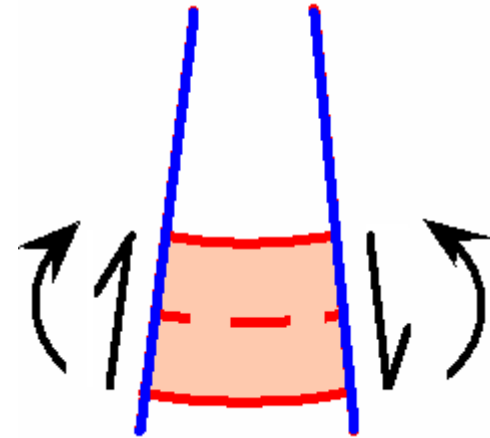


Next, turning to the beam:

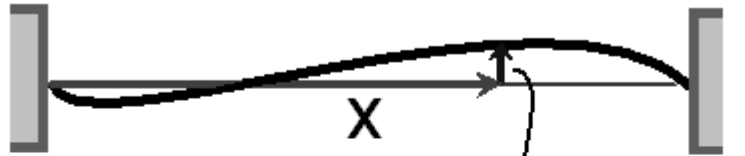


$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

"wave equation" w/
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 $u(0,t)=u(L,t)=0$

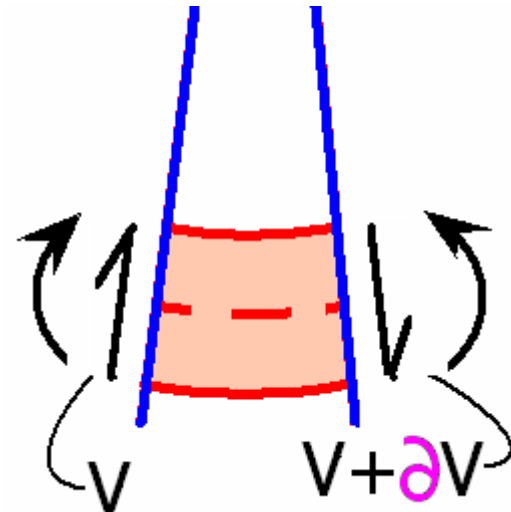


Next, turning to the beam:



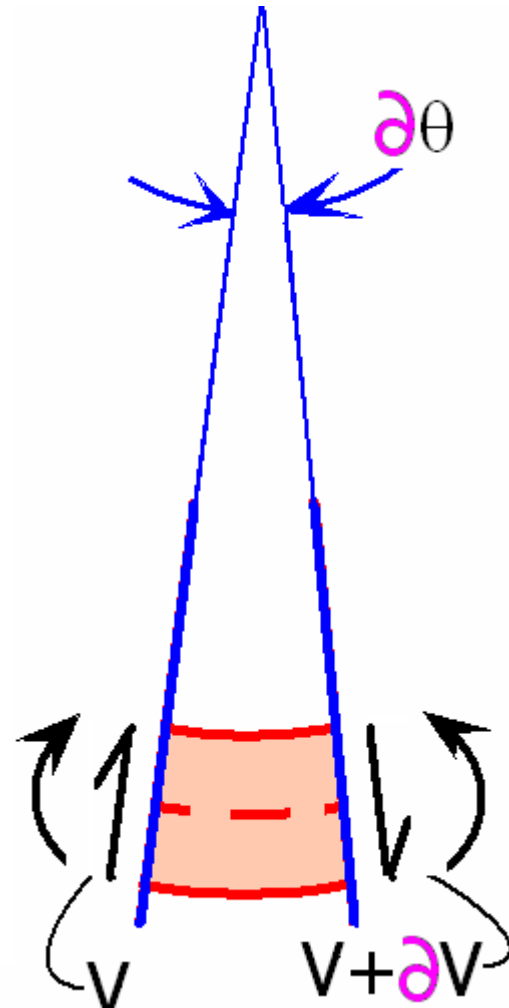
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

"wave equation" w/
boundary conditions:
 $u(0,t)=u(L,t)=0$



$$\Sigma F_{\text{vert.}} = (m/L) \Delta x \frac{\partial^2 y}{\partial t^2} = -\Delta V$$

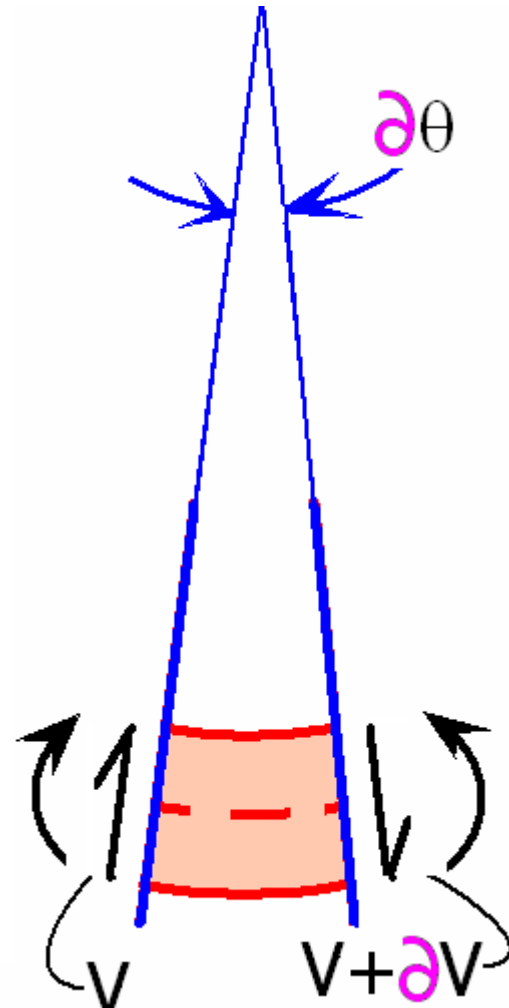
Next, turning to the beam:



$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Next, turning to the beam:

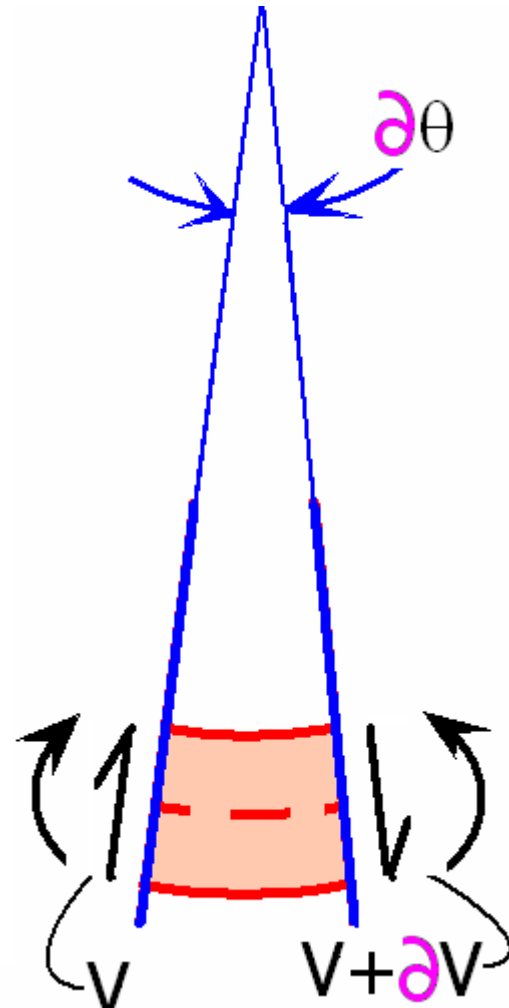
$$\rho \partial \theta = \partial x$$



$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Next, turning to the beam:

$$\rho \partial \theta = \partial x$$

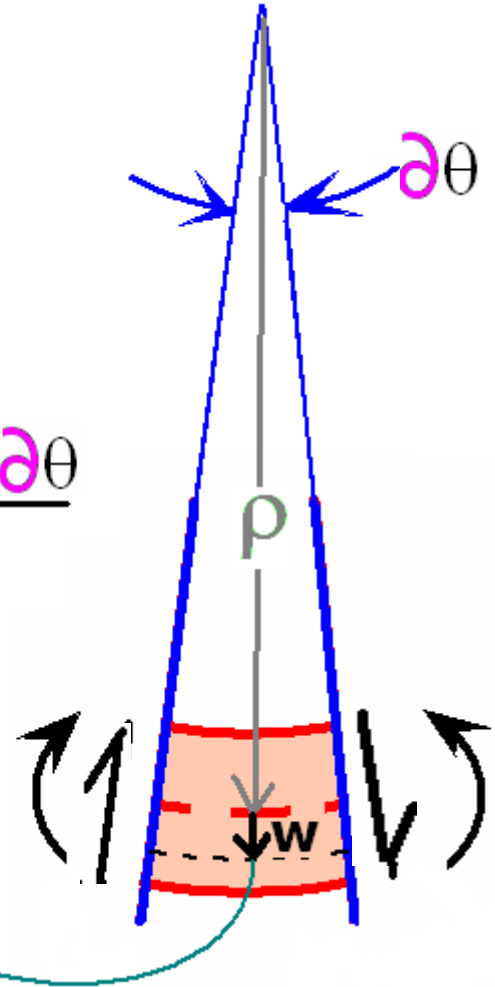


$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Next, turning to the beam:

$$\rho \partial \theta = \partial x$$

$$\varepsilon = \frac{(w + \rho) \partial \theta - \rho \partial \theta}{\rho \partial \theta}$$



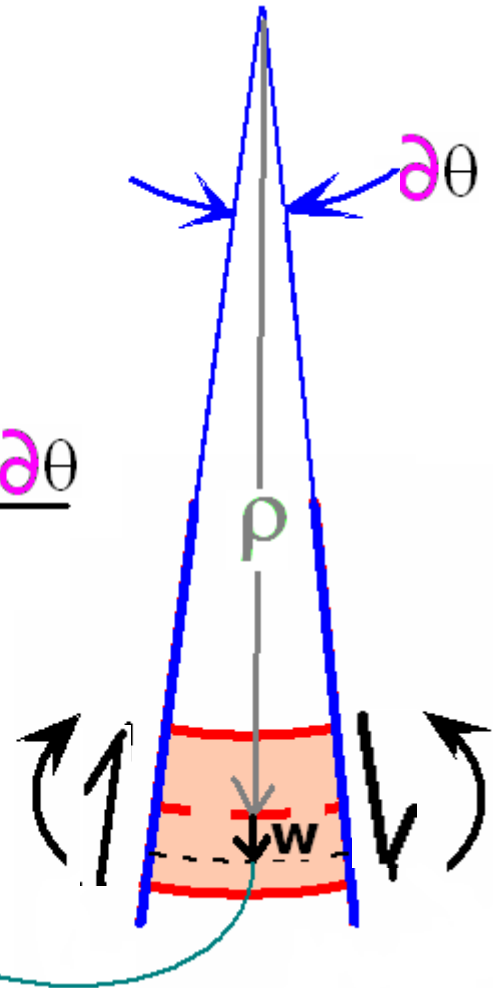
$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

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$$= \frac{w}{\rho}$$



$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Next, turning to the beam:

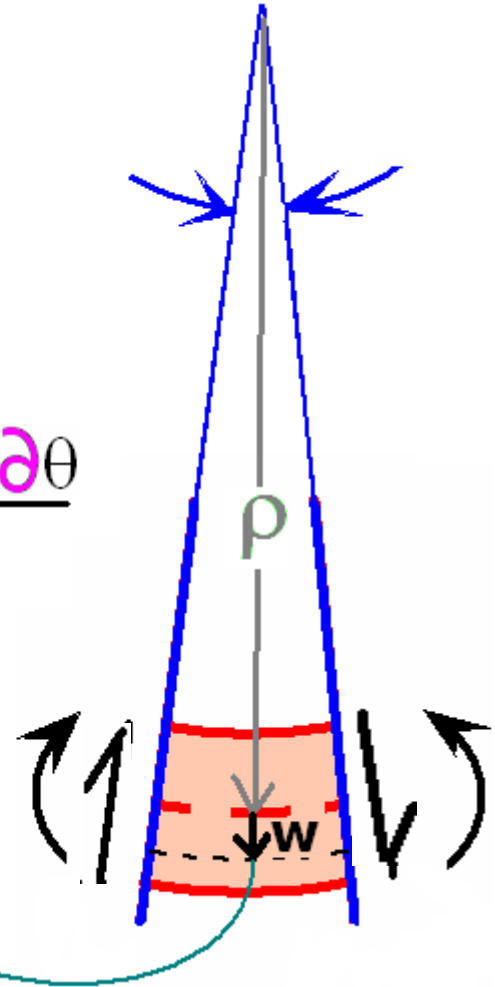
$$\rho \partial \theta = \partial x$$

$$\varepsilon = \frac{(w + \rho) \partial \theta - \rho \partial \theta}{\rho \partial \theta}$$

$$= \frac{w}{\rho}$$

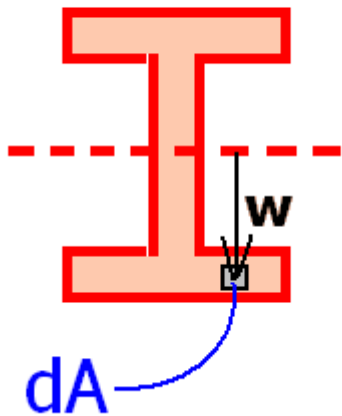
$$\sigma = E \varepsilon = E \frac{w}{\rho}$$

$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$



Next, turning to the beam:

beam cross section

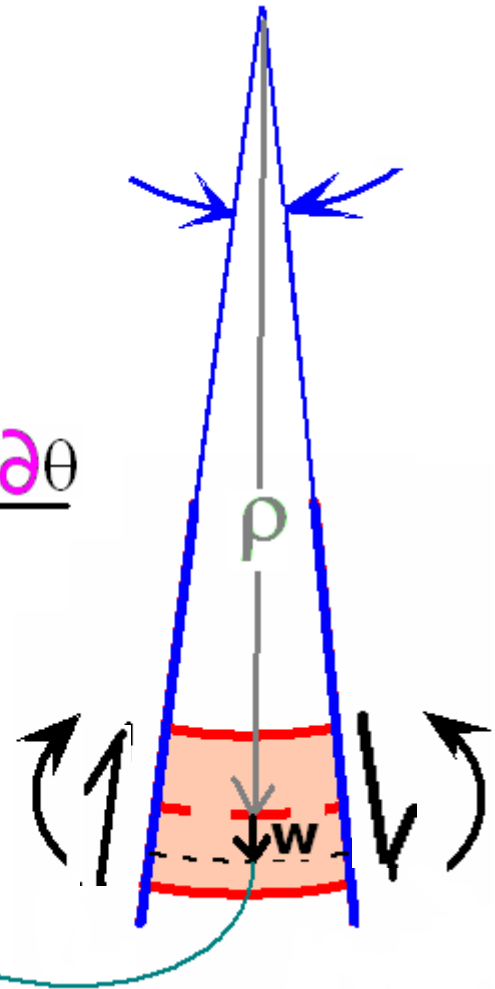


$$\rho \partial \theta = \partial x$$

$$\varepsilon = \frac{(w + \rho) \partial \theta - \rho \partial \theta}{\rho \partial \theta}$$

$$= \frac{w}{\rho}$$

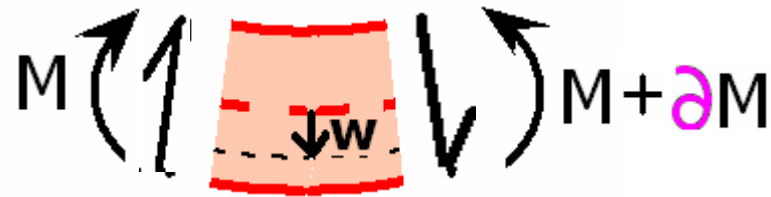
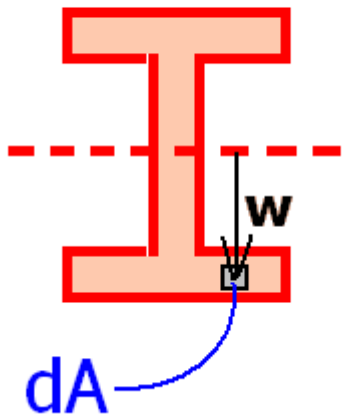
$$\sigma = E \varepsilon = E \frac{w}{\rho}$$



$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Next, turning to the beam:

beam cross section

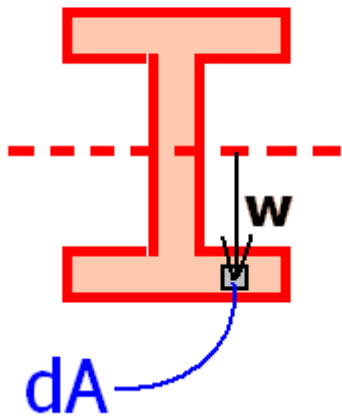


$$\sigma = E\varepsilon = E \frac{w}{\rho}$$

$$\Sigma F_{\text{vert.}} = (m/L) dx \frac{\partial^2 y}{\partial t^2} = -dV$$

Next, turning to the beam:

beam cross section



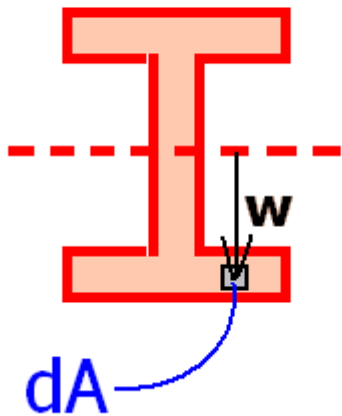
$$\iint_A w \sigma \, dA = M$$

A diagram of a beam element of length Δx . The left end is subjected to a counter-clockwise bending moment M . The right end is subjected to a clockwise bending moment $M + \Delta M$. The beam is shown with a downward deflection w at its center. The beam is colored orange with red dashed lines indicating its boundaries.

$$\Sigma F_{\text{vert.}} = (m/L) \Delta x \frac{\partial^2 y}{\partial t^2} = -\Delta V$$

Next, turning to the beam:

beam cross section



$$\iint_A w \sigma dA = M \left(\begin{array}{c} \curvearrowleft \\ \text{beam element} \\ \curvearrowright \end{array} \right) M + \partial M$$

The diagram shows a beam element of length ∂x with a trapezoidal cross-section. The left end is labeled M and the right end is labeled $M + \partial M$. A downward arrow labeled w is shown in the center of the beam element.

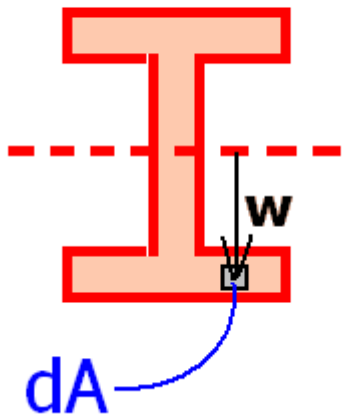
$$= \iint_A w E \frac{w}{\rho} dA$$

The diagram shows the integral $\iint_A w E \frac{w}{\rho} dA$ with purple circles around E and $\frac{w}{\rho}$, and a purple arrow pointing from the integral to the right.

$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Next, turning to the beam:

beam cross section



$$\iint_A w \sigma dA = M \left(\begin{array}{c} \curvearrowleft \\ \text{beam element} \\ \curvearrowright \end{array} \right) M + \partial M$$

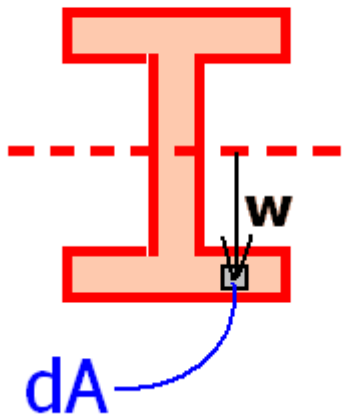
$$= \iint_A w E \frac{w}{\rho} dA = EI / \rho$$

The diagram illustrates the derivation of the beam bending equation. It shows a beam element under a bending moment M and a differential moment ∂M . The beam element is shown as a trapezoid with a downward deflection w . The equation relates the integral of the moment M over the cross-section A to the integral of the moment $M + \partial M$ over the cross-section A . The second equation shows the integral of the moment M over the cross-section A is equal to the integral of the moment $M + \partial M$ over the cross-section A , which is equal to EI / ρ .

$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Next, turning to the beam:

beam cross section



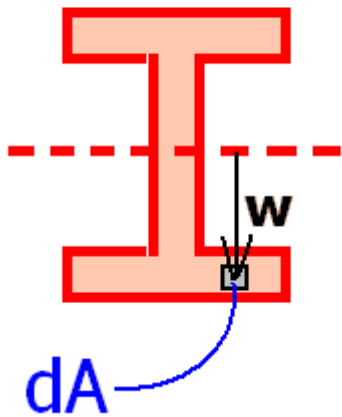
$$\iint_A w \sigma dA = M \left(\begin{array}{c} \curvearrowright \\ \text{beam element} \\ \curvearrowleft \end{array} \right) M + \partial M$$

$$= \iint_A w E \frac{w}{\rho} dA = EI / \rho \approx EI \frac{\partial^2 y}{\partial x^2}$$

$$\Sigma F_{\text{vert.}} = (m/L) \partial_x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

But note that: $v = \frac{\partial M}{\partial x}$

beam cross section



$$\iint_A w \sigma dA = M \left(\begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \right) M + \partial M$$

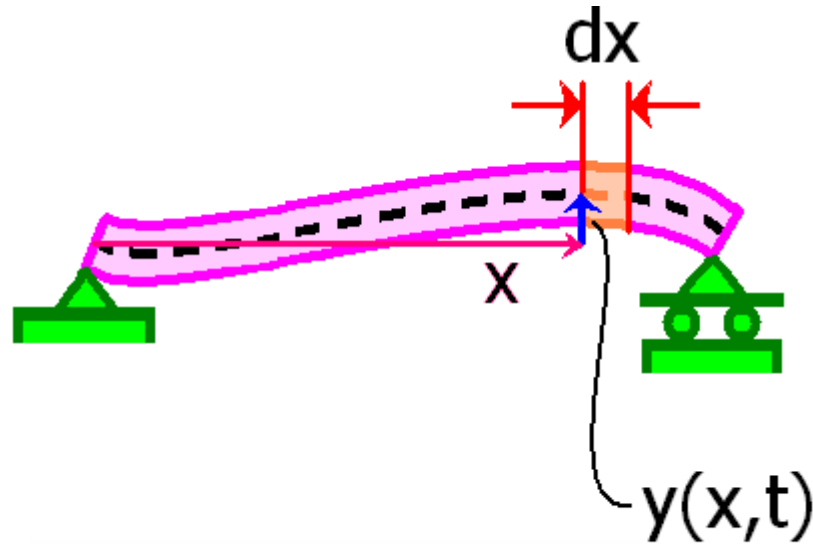
A diagram of a beam element of length ∂x . The bending moment on the left face is M (counter-clockwise) and on the right face is $M + \partial M$ (clockwise). The beam is shown with a downward deflection w at its center.

$$= \iint_A w E \frac{w}{\rho} dA = EI / \rho \approx EI \frac{\partial^2 y}{\partial x^2}$$

Diagrammatic annotations: A purple arrow points from the $\frac{w}{\rho}$ term in the second equation to the $\frac{\partial^2 y}{\partial x^2}$ term in the third equation. Another purple arrow points from the ρ term in the second equation to the $\frac{w}{\rho}$ term.

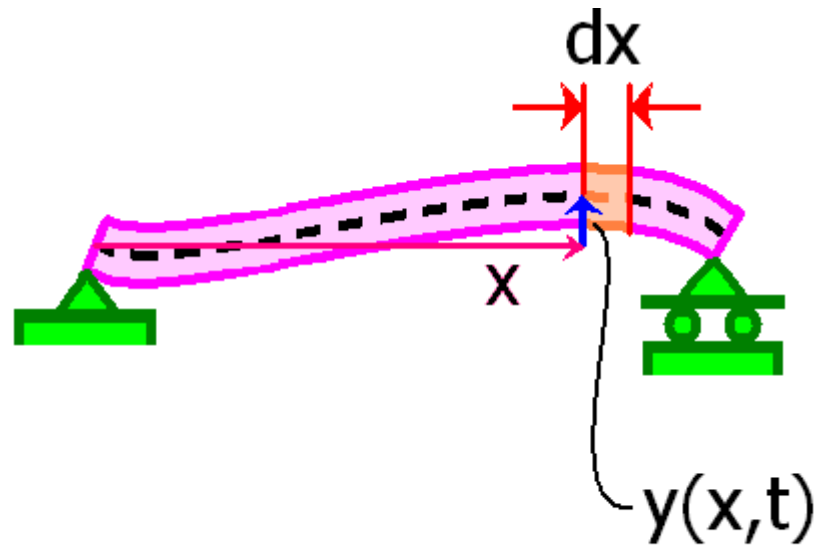
$$\Sigma F_{\text{vert.}} = (m/L) \partial x \frac{\partial^2 y}{\partial t^2} = -\partial V$$

Putting all of this together:



$$-(m/L) \frac{\partial}{\partial x} \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x} \frac{\partial M}{\partial x}$$

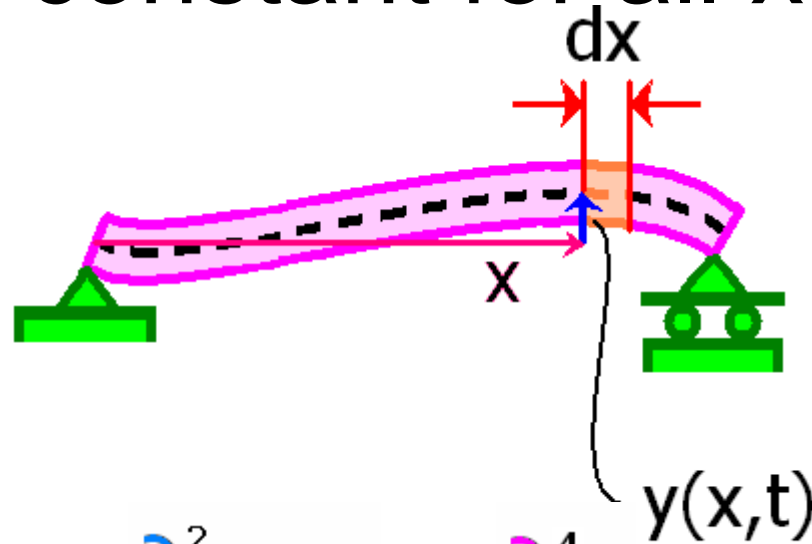
Putting all of this together:



$$-(m/L) \frac{\partial^2 y}{\partial t^2} = \frac{\partial M}{\partial x}$$

$$-(m/L) \frac{\partial^2 y}{\partial t^2} \approx \frac{\partial^2}{\partial x^2} \left[EI \frac{\partial^2 y}{\partial x^2} \right]$$

If the beam's cross section is the same across its length, such that EI is a constant for all x , then,

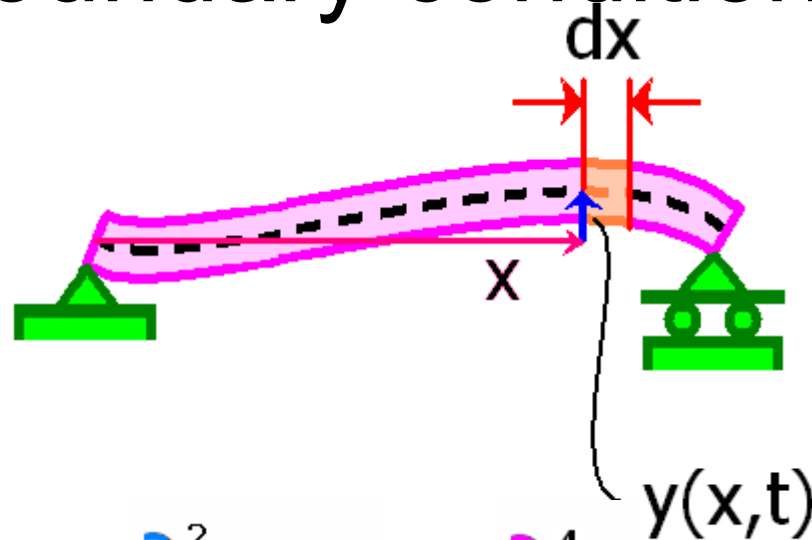


$$\frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^4 y}{\partial x^4}$$

where

$$a^2 = EI/m$$

Because we have a fourth spatial derivative, we look for four boundary conditions



$$\frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^4 y}{\partial x^4}$$

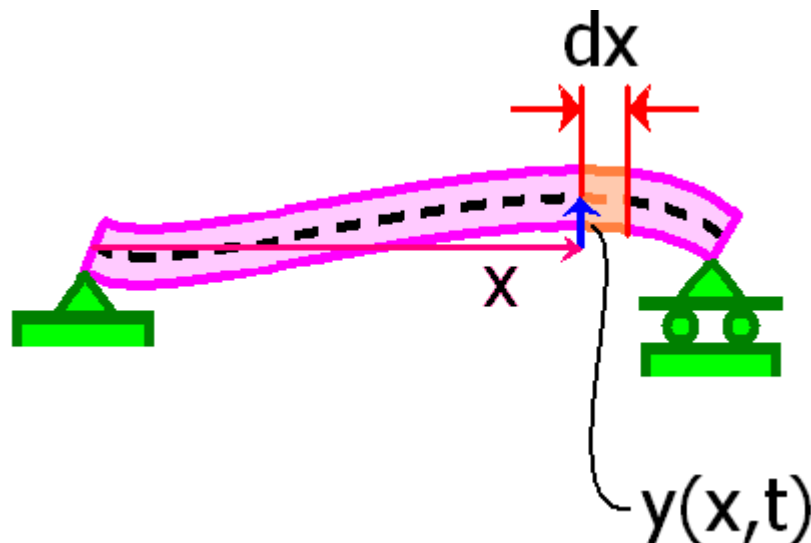
where

$$a^2 = EIL/m$$

Two of these are called “geometric boundary conditions”. They pertain to zero displacement at the two extreme ends:

$$y(0,t)=0$$

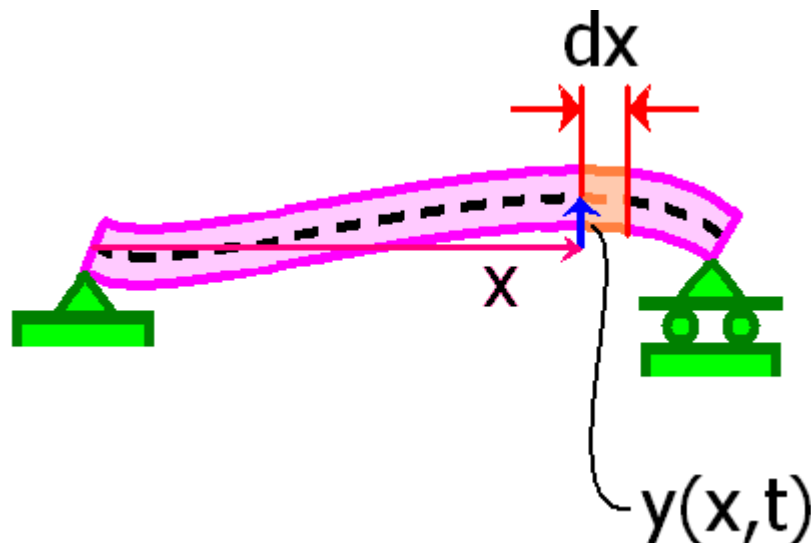
$$y(L,t)=0$$



The other two are identified with zero bending moment, considered “natural boundary conditions”, at each of the two ends. They may be written:

$$M(0,t)=EIy''(0,t)=0$$

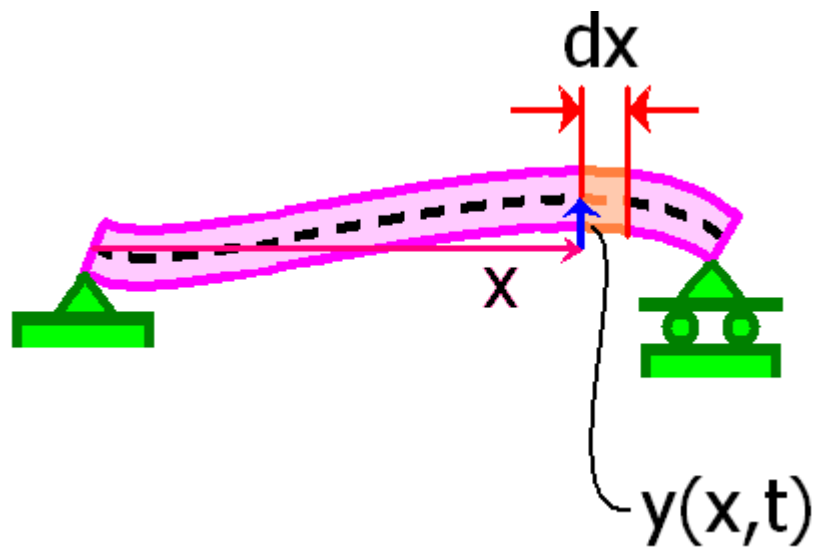
$$M(L,t)=EIy''(L,t)=0$$



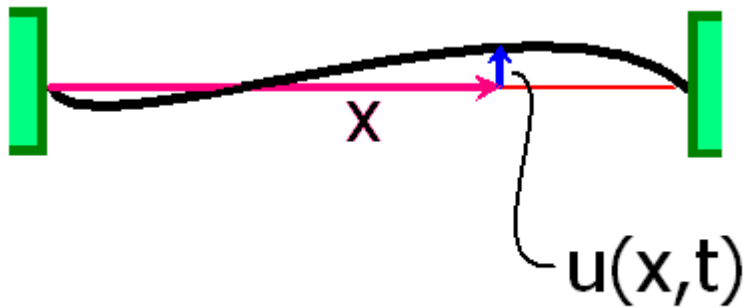
or:

$$y''(0,t)=0$$

$$y''(L,t)=0$$



Our two systems are governed according to:

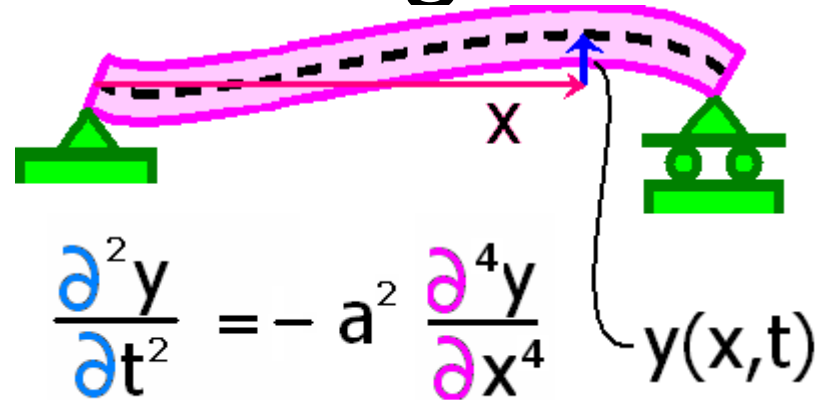


$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where

$$a^2 = TL/m$$

$$u(0,t)=u(L,t)=0$$



$$\frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^4 y}{\partial x^4}$$

where

$$a^2 = EIL/m$$

$$y(0,t)=0$$

$$y''(0,t)=0$$

$$y(L,t)=0$$

$$y''(L,t)=0$$

HW 37

Due Dec. 5, 2005

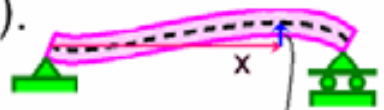
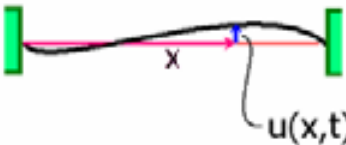
Problem A: We consider the question of solving for the natural frequencies of the taut cable and the simply supported Euler-Bernoulli beam. Assuming for each of the two o.d.e.s on the right a solution form $X(x)=Ae^{rx}$, determine:

(a) A characteristic polynomial of second degree whose roots in r (denoted here as $r_1=i\omega/a$, $r_2=-i\omega/a$) admit a solution to the taut cable's (lower left of the two boxes) o.d.e.

(b) A characteristic polynomial of fourth degree whose 4 roots in r admit a solution to the simply supported beam's o.d.e.

Q: But how do we get ω^2 ?

A: Look at $X(x)$.


$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^4 y}{\partial x^4}$$

$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

$$\frac{d^2 X}{dx^2} + \frac{\omega^2}{a^2} X = 0$$

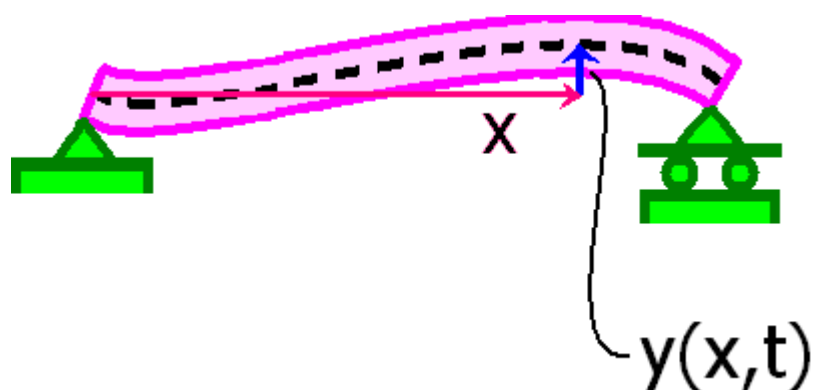


Problem B: Turning our attention to the beam: We may write the four roots of the characteristic equation: $r_1 = \sqrt{\omega/a}$, $r_2 = -\sqrt{\omega/a}$, $r_3 = i\sqrt{\omega/a}$, $r_4 = -i\sqrt{\omega/a}$. Show that the general solution:

$$X(x) = A_1 \exp(r_1 x) + A_2 \exp(r_2 x) + A_3 \exp(r_3 x) + A_4 \exp(r_4 x)$$

may also be written

$$X(x) = C_1 \cosh[\sqrt{\omega/a}x] + C_2 \sinh[\sqrt{\omega/a}x] + C_3 \cos[\sqrt{\omega/a}x] + C_4 \sin[\sqrt{\omega/a}x]$$



Let the dependent variable,
 y or u , be $X(x)T(t)$

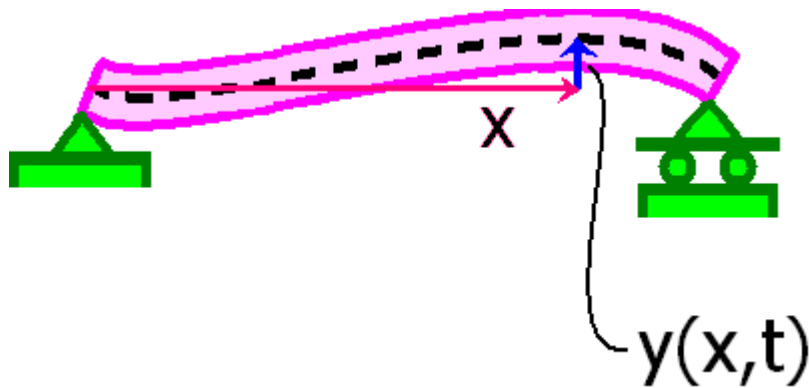
$$\frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^4 y}{\partial x^4}$$

where

$$a^2 = EI/L^3$$

$$y(0,t)=0 \quad M(0,t)=EIy''(0,t)=0$$

$$y(L,t)=0 \quad M(L,t)=EIy''(L,t)=0$$



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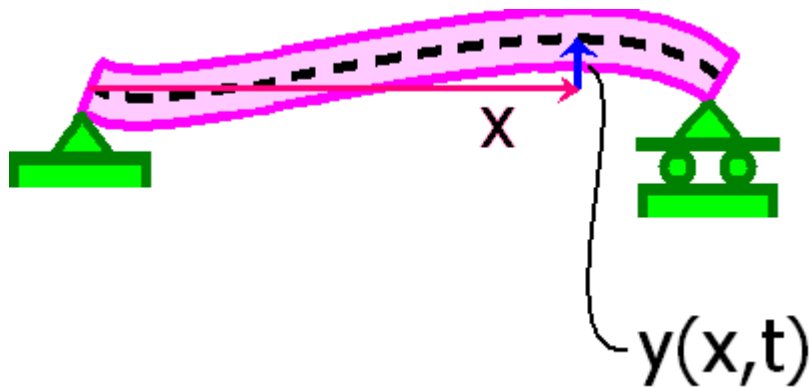
$$X(x) \frac{d^2 T}{dt^2} = -a^2 T(t) \frac{d^4 X}{dx^4}$$

where

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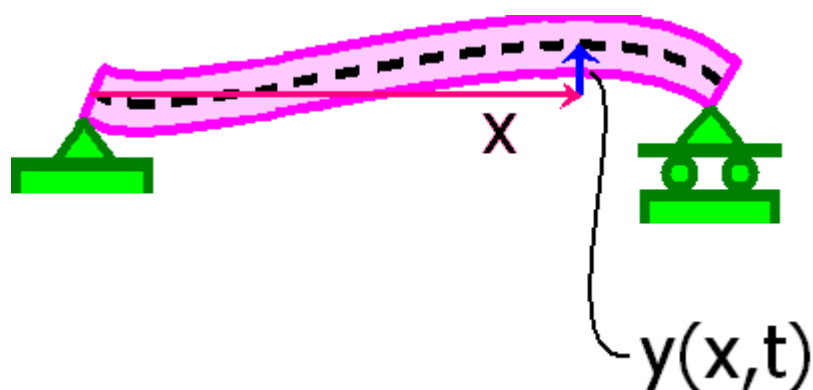
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where

$$a^2 = EI/m$$

$$\boxed{\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0}$$

$$y(0,t)=0 \quad M(0,t)=EIy''(0,t)=0$$

$$y(L,t)=0 \quad M(L,t)=EIy''(L,t)=0$$

$$\begin{array}{l} y(x,t) = \\ X(x)T(t) \end{array} \Rightarrow \begin{array}{l} X(0)=0 \\ X(L)=0 \\ X''(0)=0 \\ X''(L)=0 \end{array}$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

$$X(x) = C_1 \cosh[\sqrt{\omega/a}x] + C_2 \sinh[\sqrt{\omega/a}x] + C_3 \cos[\sqrt{\omega/a}x] + C_4 \sin[\sqrt{\omega/a}x]$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

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$$X(0)=0$$

$$X(L)=0$$

$$X''(0)=0$$

$$X''(L)=0$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

$$X(0)=0$$

$$X(x) = C_1 \cosh[\sqrt{\omega/a}x] + C_2 \sinh[\sqrt{\omega/a}x] + C_3 \cos[\sqrt{\omega/a}x] + C_4 \sin[\sqrt{\omega/a}x]$$

$$C_1 + C_3 = 0$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

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$$C_1 + C_3 = 0$$

$$C_1 - C_3 = 0$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

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$$\left. \begin{array}{l} C_1 + C_3 = 0 \\ C_1 - C_3 = 0 \end{array} \right\} C_1 = C_3 = 0$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

$$X(x) = C_1 \cosh[\sqrt{\omega/a}x] + C_2 \sinh[\sqrt{\omega/a}x] + C_3 \cos[\sqrt{\omega/a}x] + C_4 \sin[\sqrt{\omega/a}x] \quad X(L)=0$$

$$C_2 \sinh \sqrt{\omega/a} L + C_4 \sin \sqrt{\omega/a} L = 0$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

$$X(x) = C_1 \cosh[\sqrt{\omega/a}x] + C_2 \sinh[\sqrt{\omega/a}x] + C_3 \cos[\sqrt{\omega/a}x] + C_4 \sin[\sqrt{\omega/a}x]$$

$$X''(L) = 0$$

$$C_2 \sinh \sqrt{\omega/a} L + C_4 \sin \sqrt{\omega/a} L = 0$$

$$[\omega/a][C_2 \sinh \sqrt{\omega/a} L - C_4 \sin \sqrt{\omega/a} L] = 0$$



$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{a^2} X = 0$$

$$X(x) = C_1 \cosh[\sqrt{\omega/a}x] + C_2 \sinh[\sqrt{\omega/a}x] + C_3 \cos[\sqrt{\omega/a}x] + C_4 \sin[\sqrt{\omega/a}x]$$

$$X''(L) = 0$$

$$\begin{bmatrix} \sinh \sqrt{\omega/a} L & \sin \sqrt{\omega/a} L \\ \sinh \sqrt{\omega/a} L & -\sin \sqrt{\omega/a} L \end{bmatrix} \begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



To avoid the trivial solution,
select ω such that:

$$\begin{bmatrix} \sinh \sqrt{\omega/a} L & \sin \sqrt{\omega/a} L \\ \sinh \sqrt{\omega/a} L & -\sin \sqrt{\omega/a} L \end{bmatrix} \begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



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$$\begin{vmatrix} \sinh \sqrt{\omega/a} L & \sin \sqrt{\omega/a} L \\ \sinh \sqrt{\omega/a} L & -\sin \sqrt{\omega/a} L \end{vmatrix} = 0$$

$$2 \sinh \sqrt{\omega/a} L \sin \sqrt{\omega/a} L = 0$$

For illustration take: $a=1$ $L=1$



To avoid the trivial solution,
select ω such that:

$$\begin{vmatrix} \sinh \sqrt{\omega/a} L & \sin \sqrt{\omega/a} L \\ \sinh \sqrt{\omega/a} L & -\sin \sqrt{\omega/a} L \end{vmatrix} = 0$$

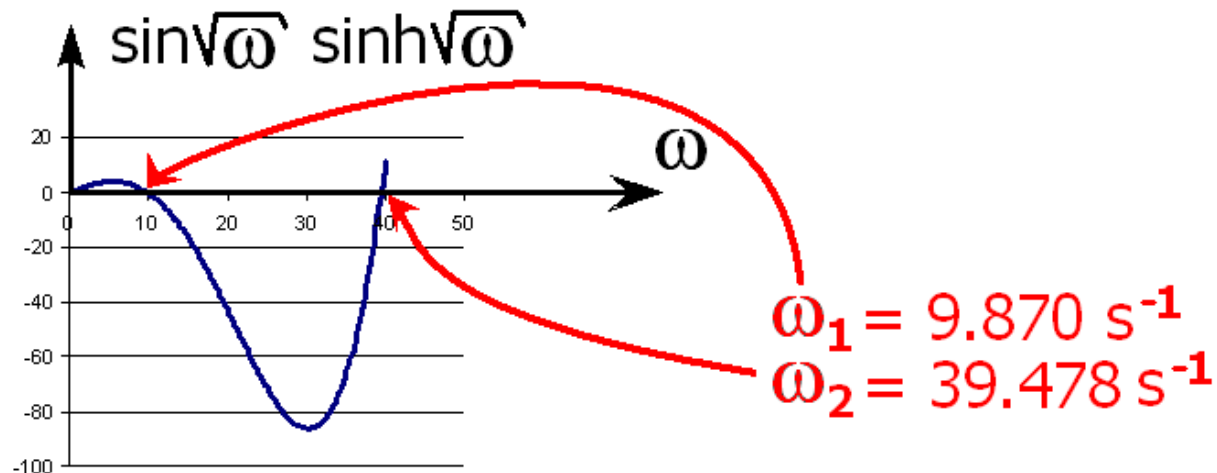
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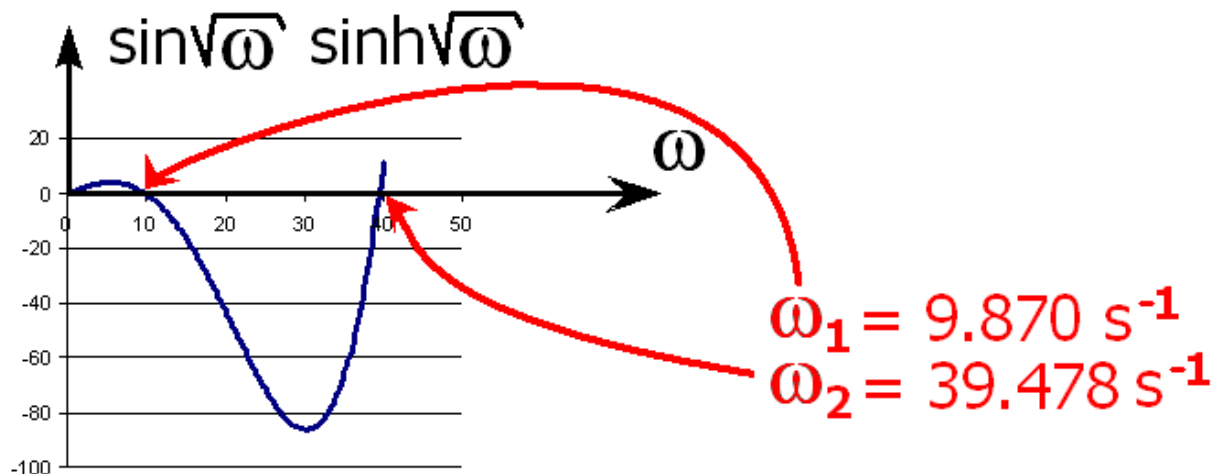
For illustration take: $a=1$ $L=1$



Either of the previous equations can be applied to relate C_2 and C_4 :

$$C_2 \sinh \sqrt{\omega/a} L + C_4 \sin \sqrt{\omega/a} L = 0$$

$$[\omega/a][C_2 \sinh \sqrt{\omega/a} L - C_4 \sin \sqrt{\omega/a} L] = 0$$

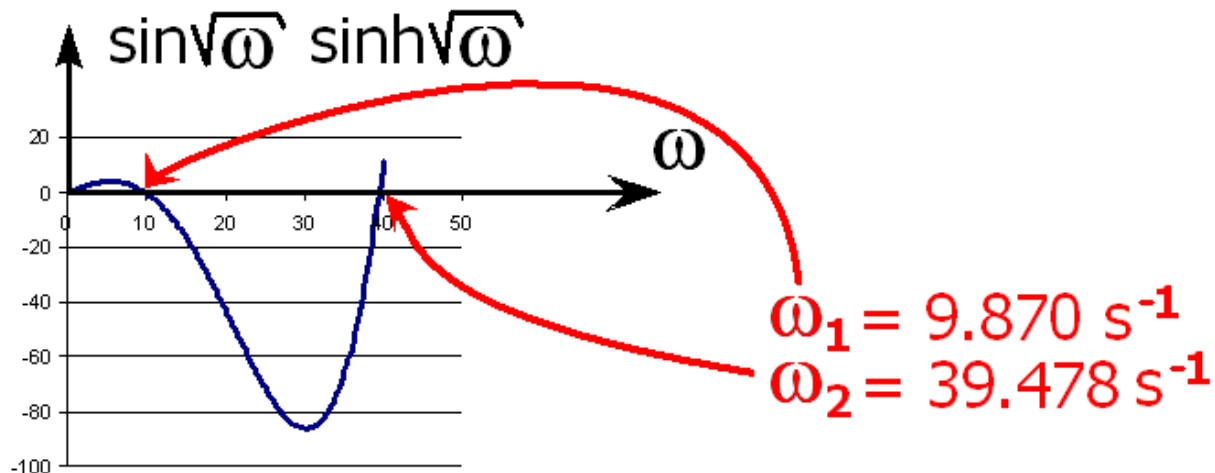


For illustration take: $a=1$ $L=1$



Let's use the first of the two
with the first natural
frequency:

$$C_2 \sinh \sqrt{9.870} + C_4 \sin \sqrt{9.870} = 0$$



For illustration take: $a=1$ $L=1$



Let's use the first of the two
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frequency:

$$C_2 \sinh \sqrt{9.870} + C_4 \sin \sqrt{9.870} = 0$$

A purple arrow points from the $\sqrt{9.870}$ term in the sine function to a superscript 0, indicating that the argument of the sine function is zero.

For illustration take: $a=1$ $L=1$



Let's use the first of the two
with the first natural
frequency:

$$C_2 \sinh \sqrt{9.870} + C_4 \sin \sqrt{9.870} = 0$$

A purple arrow points from the $\sqrt{9.870}$ term in the sine function to a '0' at the top right of the equation.

$$\therefore C_2 = 0$$

For illustration take: $a=1$ $L=1$



Let's use the first of the two
with the first natural
frequency:

$$C_2 \sinh \sqrt{9.870} + C_4 \sin \sqrt{9.870} = 0$$

A purple arrow points from the term $C_4 \sin \sqrt{9.870}$ to a '0' at the top right of the equation.

$$\therefore C_2 = 0$$

$$X(x) = C_1 \cosh[\sqrt{\omega/a}x] + C_2 \sinh[\sqrt{\omega/a}x] + C_3 \cos[\sqrt{\omega/a}x] + C_4 \sin[\sqrt{\omega/a}x]$$

For illustration take: $a=1$ $L=1$



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$$C_2 \sinh \sqrt{9.870} + C_4 \sin \sqrt{9.870} = 0$$

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$$\therefore C_2 = 0$$

$$X_1(x) = C_4 \sin \sqrt{9.870} x$$

For illustration take: $a=1$ $L=1$



Let's use the first of the two
with the first natural
frequency:

recall: $y(x,t)=X(x)T(t)$

$$T(t)=c_1 \cos \omega_1 t + d_1 \sin \omega_1 t$$

$$X_1(x) = C_4 \sin \sqrt{9.870} x$$

For illustration take: $a=1$ $L=1$

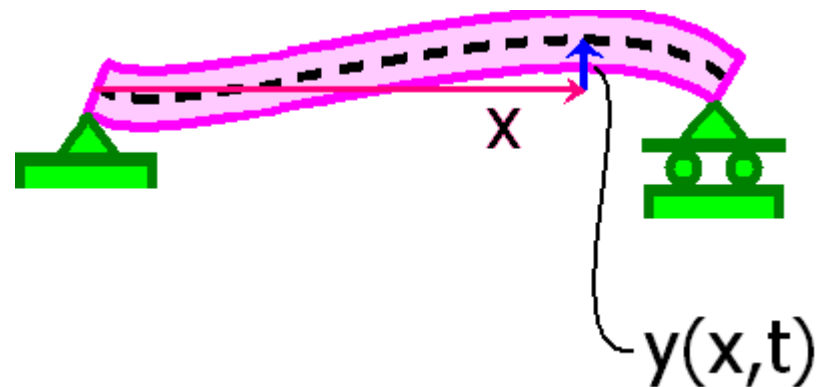


Let's use the first of the two
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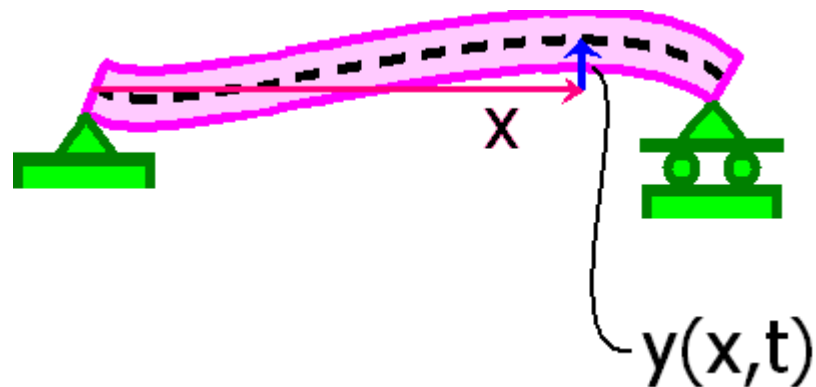
$$T(t)=c_1 \cos \omega_1 t + d_1 \sin \omega_1 t$$

$$X_1(x) = C_4 \sin \sqrt{9.870} x$$

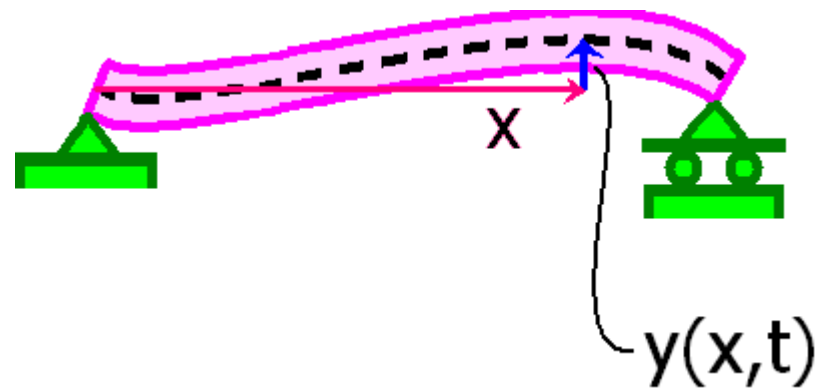


recall: $y(x,t) = X(x)T(t)$
 $T(t) = c_1 \cos \omega_1 t + d_1 \sin \omega_1 t$

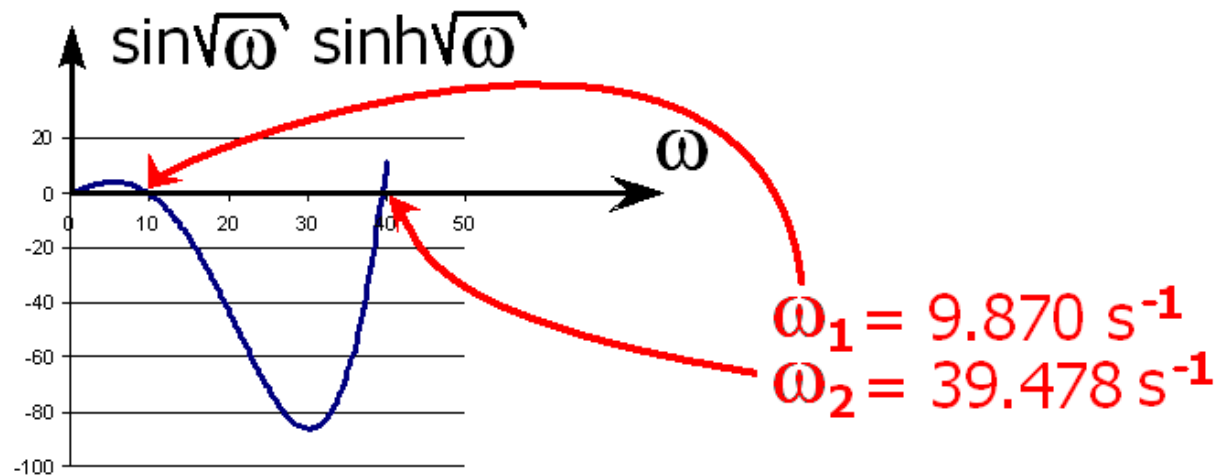
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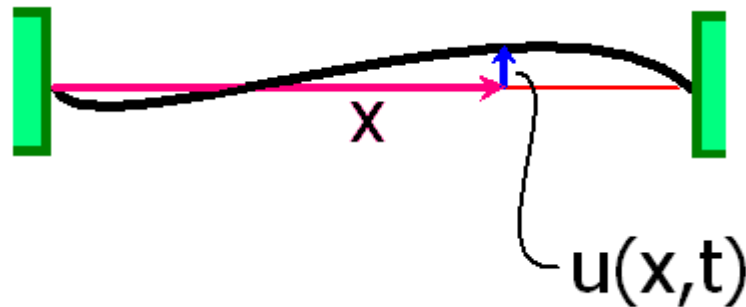
recall: $y(x,t) = \sum X_i(x) T_i(t)$
 $T_i(t) = c_i \cos \omega_i t + d_i \sin \omega_i t$



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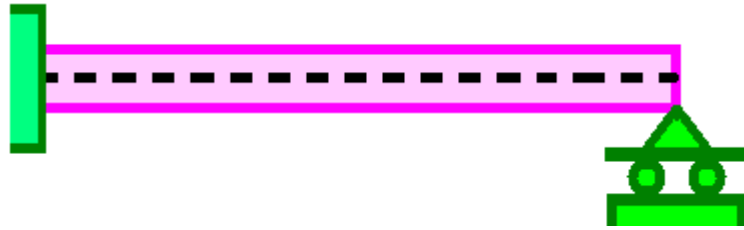


The simply supported beam turns out to have the same mode shapes as the taut cable.

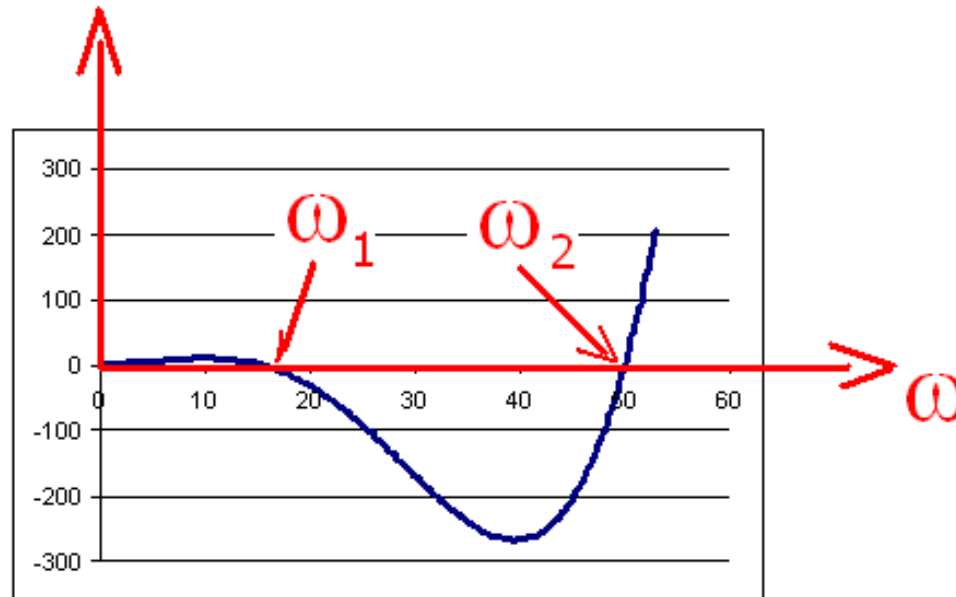


For a beam having a different type of support, this is not the case.

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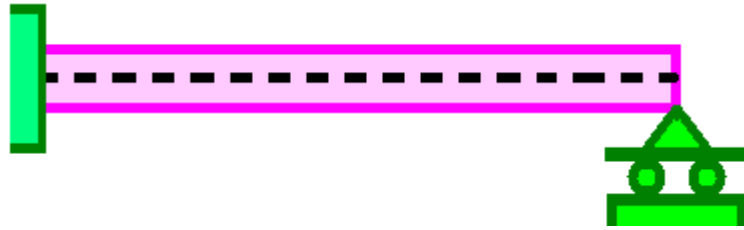


Last homework

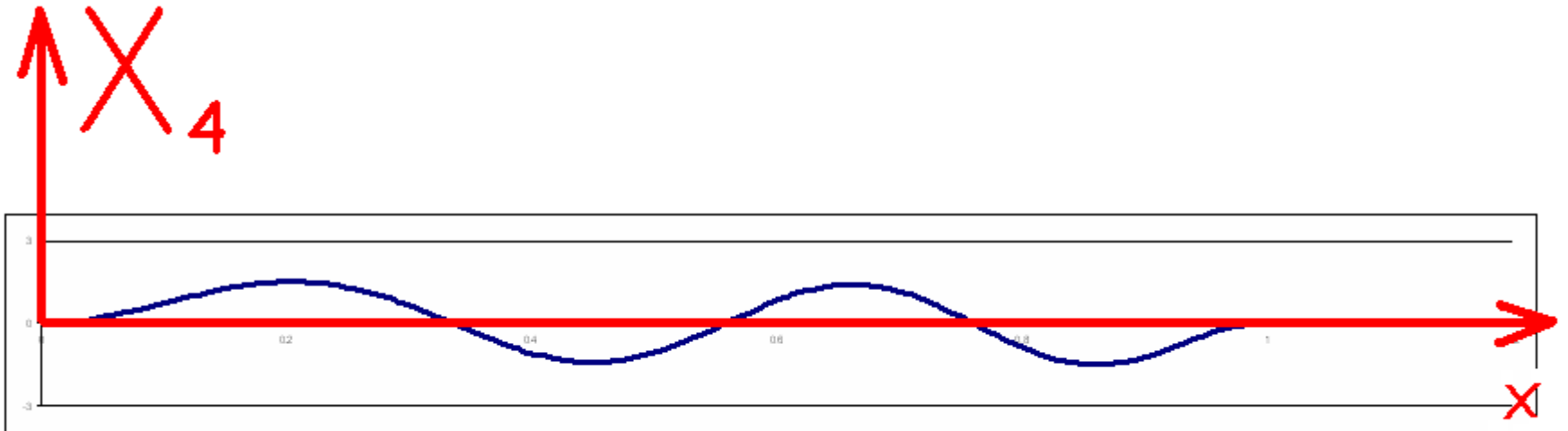


Cantilever beam w/ pin support at right end: $X(0)=0$; $X'(0)=0$; $X(L)=0$; $X''(L)=0$.

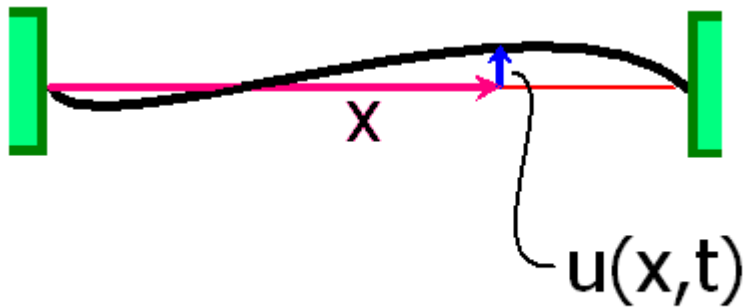
Fourth natural frequency
is 104.25 s^{-1}



Cantilever beam w/ pin support at right
end: $X(0)=0$; $X'(0)=0$; $X(L)=0$; $X''(L)=0$.



Our two systems are governed according to:

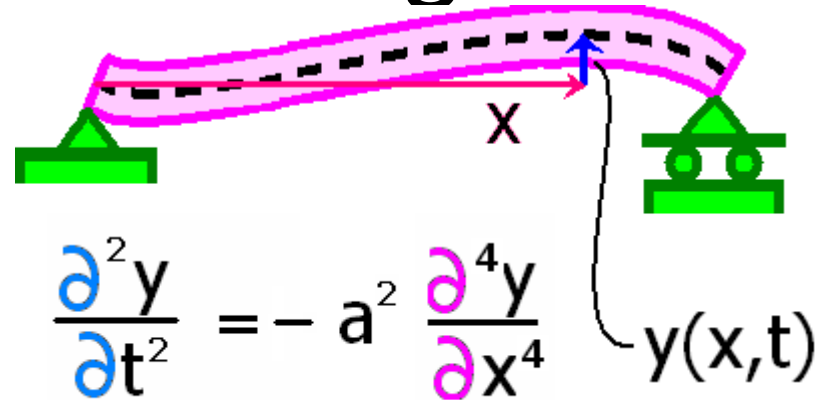


$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where

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$$u(0,t)=u(L,t)=0$$



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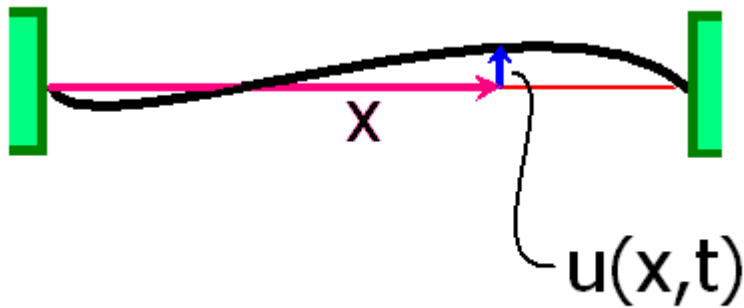
$$y(0,t)=0$$

$$y''(0,t)=0$$

$$y(L,t)=0$$

$$y''(L,t)=0$$

Consider the vibrating cable on the left

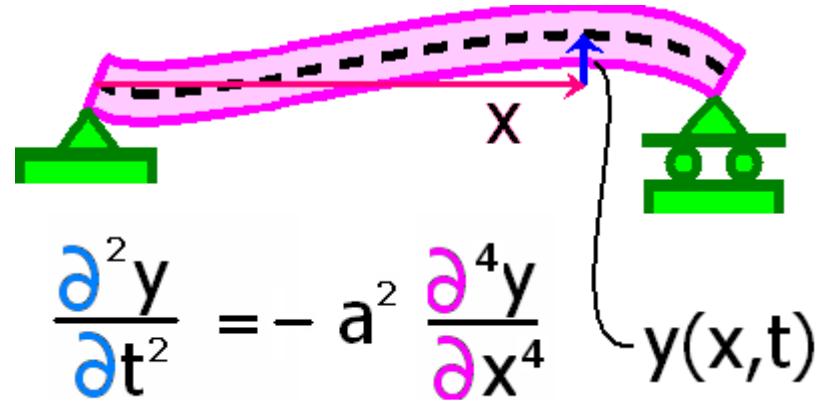


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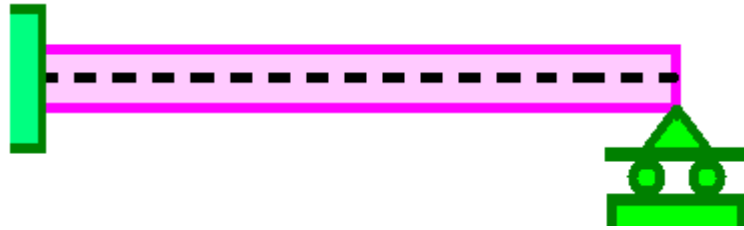
$$y(0,t)=0$$

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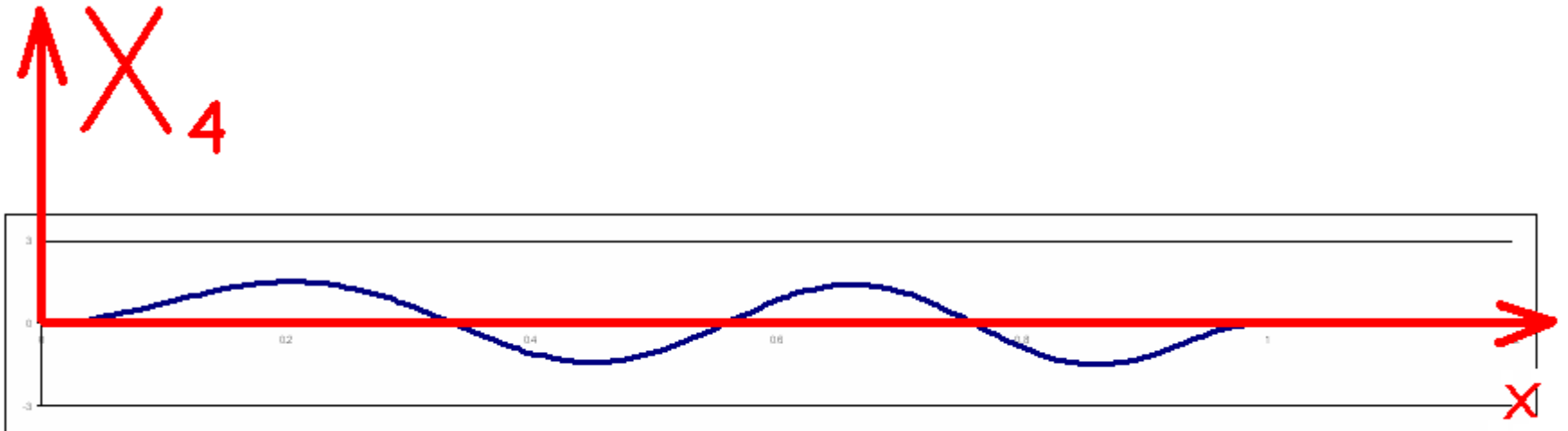
$$y(L,t)=0$$

$$y''(L,t)=0$$

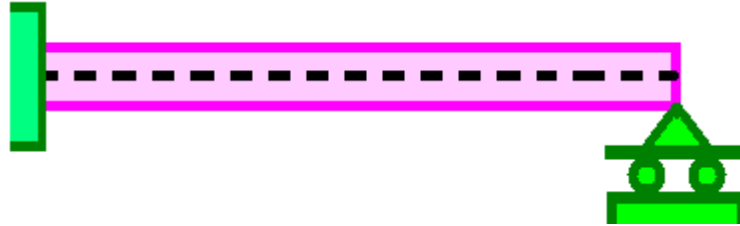
Fourth natural frequency
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Cantilever beam w/ pin support at right
end: $X(0)=0$; $X'(0)=0$; $X(L)=0$; $X''(L)=0$.

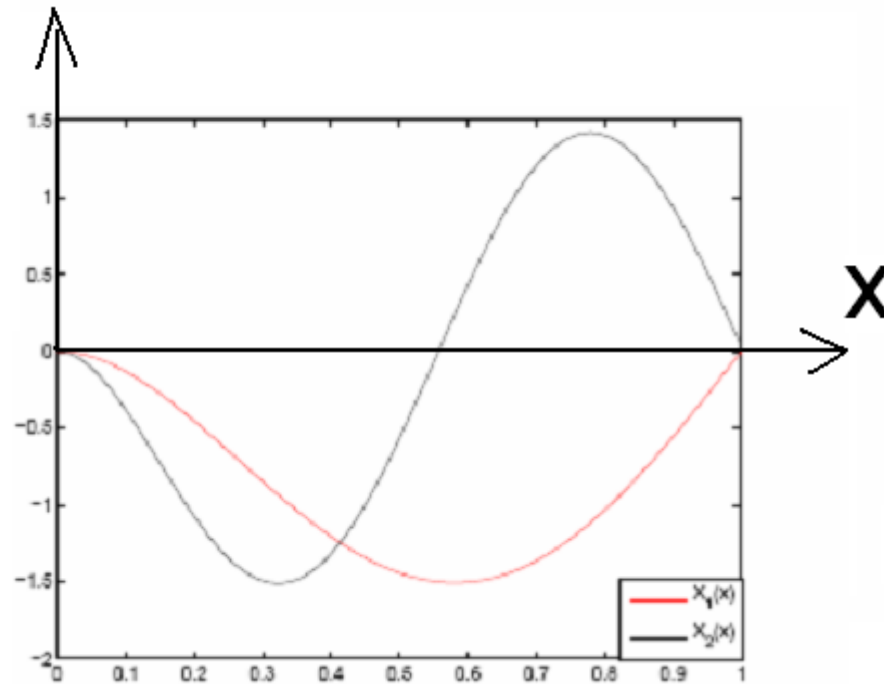


First two mode shapes

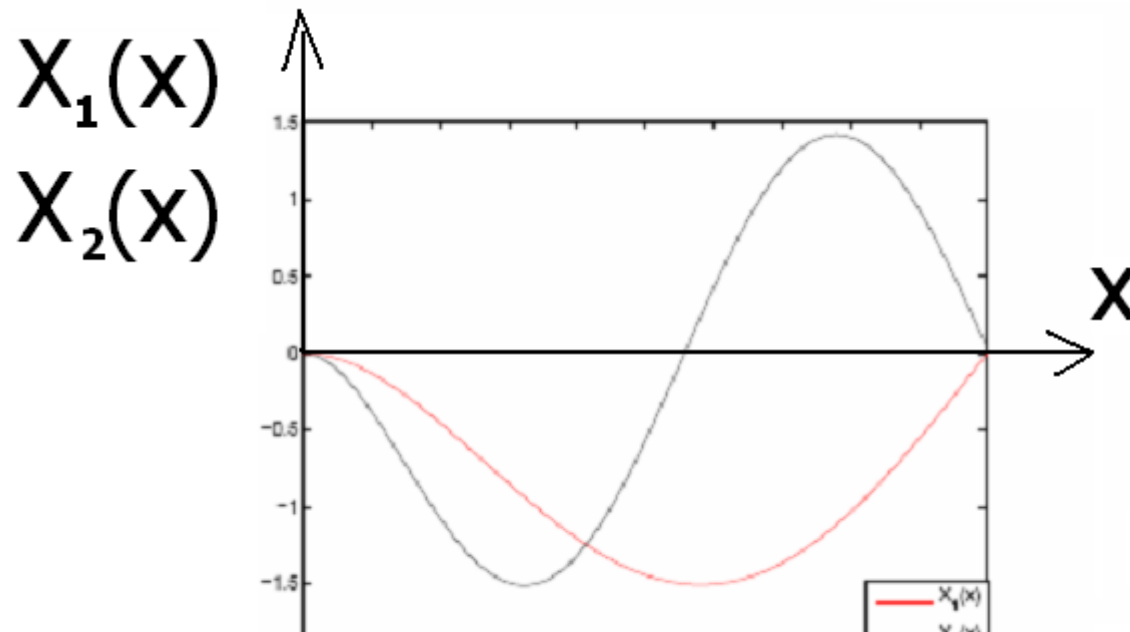
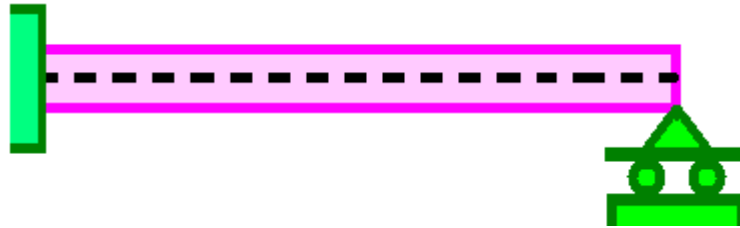


$X_1(x)$

$X_2(x)$

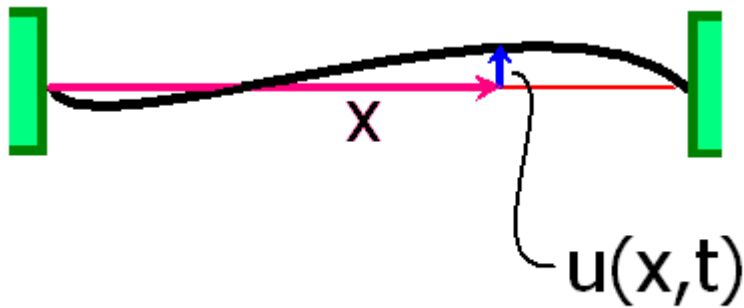


First two mode shapes



Cantilever beam w/ pin support at right end: $X(0)=0$; $X'(0)=0$; $X(L)=0$; $X''(L)=0$.

Consider the vibrating cable on the left



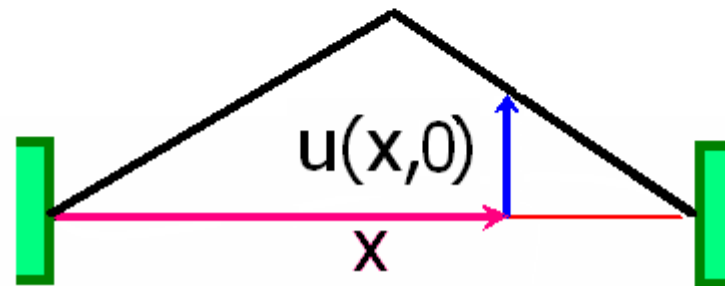
Initial conditions are comprised of a given $u(x,0)$ and an assumption of zero initial velocity, i.e. $u_t(x,0)=0$.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

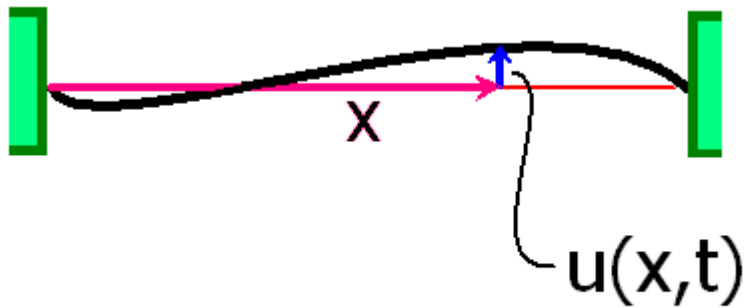
where

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Consider the vibrating cable on the left



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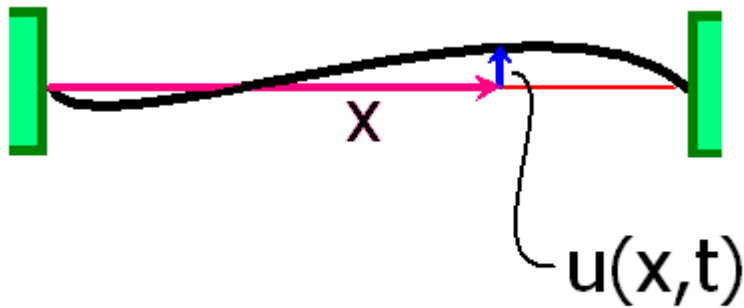
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Consider the vibrating cable on the left



Initial conditions are comprised of a given $u(x, 0)$ and an assumption of zero initial velocity, i.e. $u_t(x, 0) = 0$.

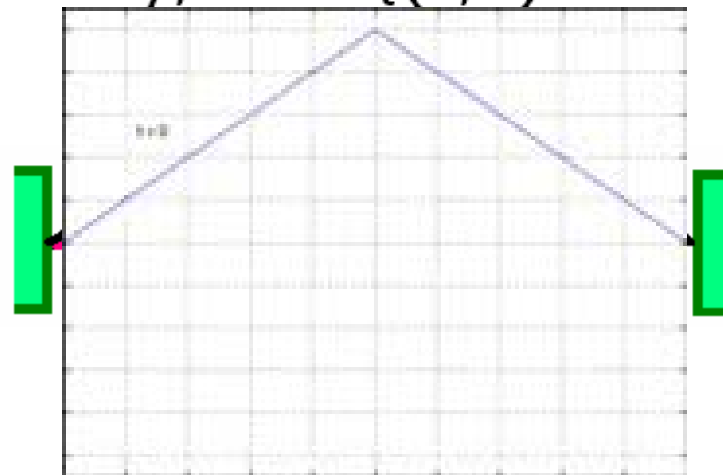
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

We assumed a solution:

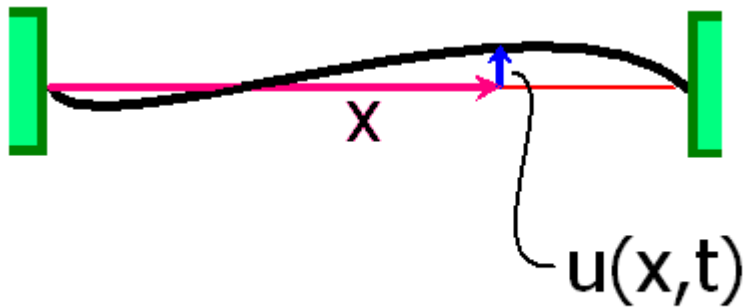
$$u(x, t) = X(x)T(t)$$

$$T(t) = c \cos \omega_n t + d \sin \omega_n t$$

$$X(x) = \sin n \pi x / L$$



Consider the vibrating cable on the left

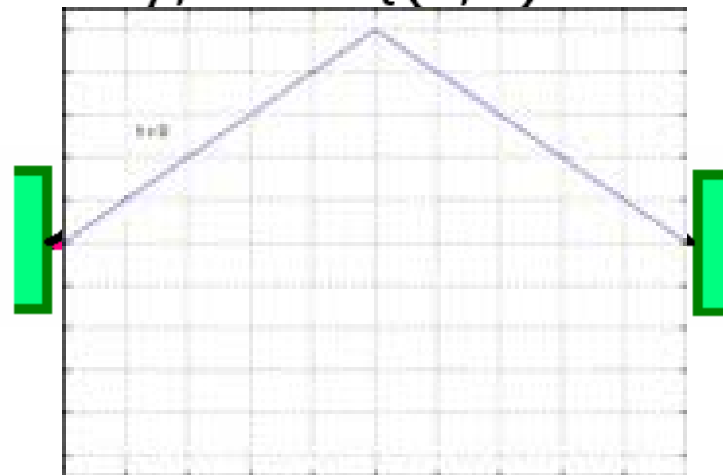


Initial conditions are comprised of a given $u(x,0)$ and an assumption of zero initial velocity, i.e. $u_t(x,0)=0$.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

We assumed a solution:

$$u(x,t)=X(x)T(t)$$

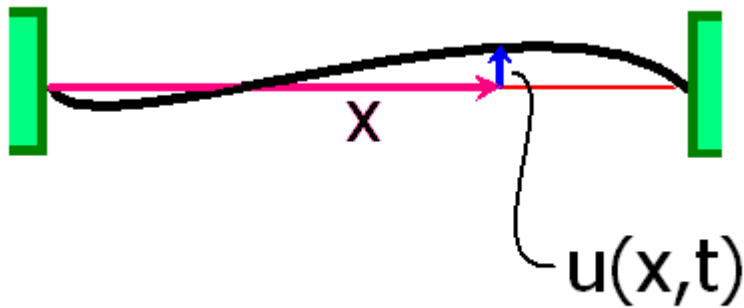


$$T(t)=c \cos \omega_n t + d \sin \omega_n t$$

$$X(x) = \sin n \pi x/L$$

$$n = 1, 2, 3 \dots$$

Consider the vibrating cable on the left



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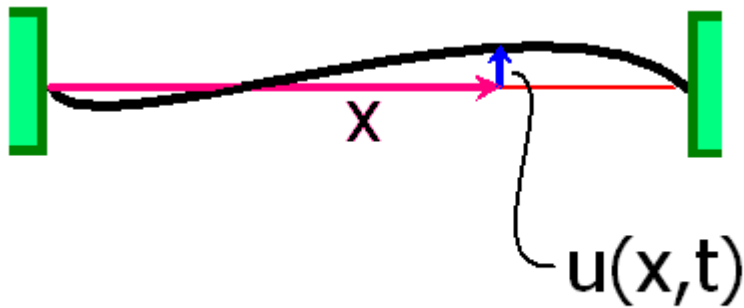
$$T_n(t) = c_n \cos \omega_n t + d_n \sin \omega_n t$$

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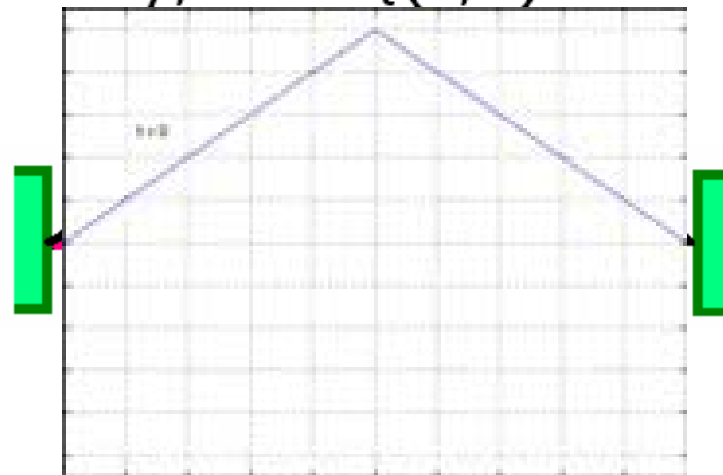
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Due to linearity, superposition:

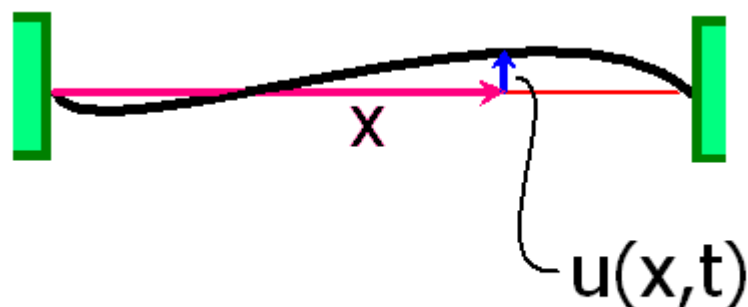
$$u(x,t) = \sum_n X_n(x) T_n(t)$$

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$$X_n(x) = \sin n \pi x / L$$



$$n = 1, 2, 3 \dots$$



Think of our init.
cond'n: $u(x,0)$

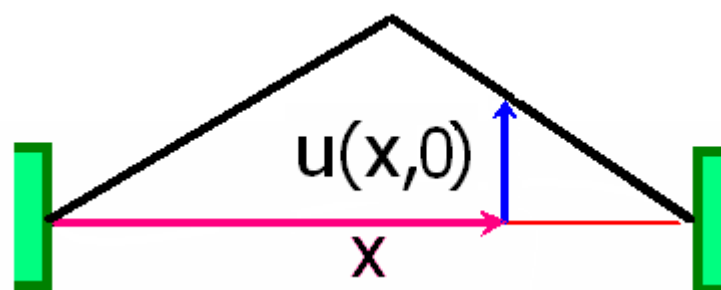
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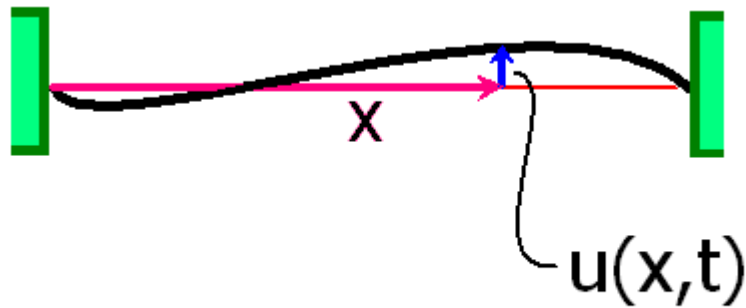
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$$n = 1, 2, 3 \dots$$

$$u(x,0) = \sum_n c_n \sin n \pi x/L$$

$$= \sum_n c_n X_n(x)$$



Think of our init.
cond'n: $u(x,0)$

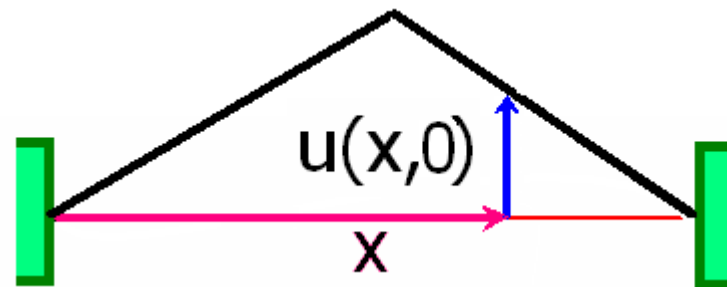
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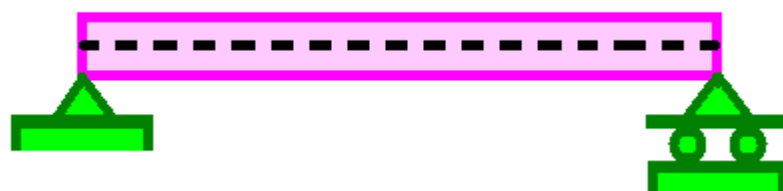
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$$n = 1, 2, 3 \dots$$

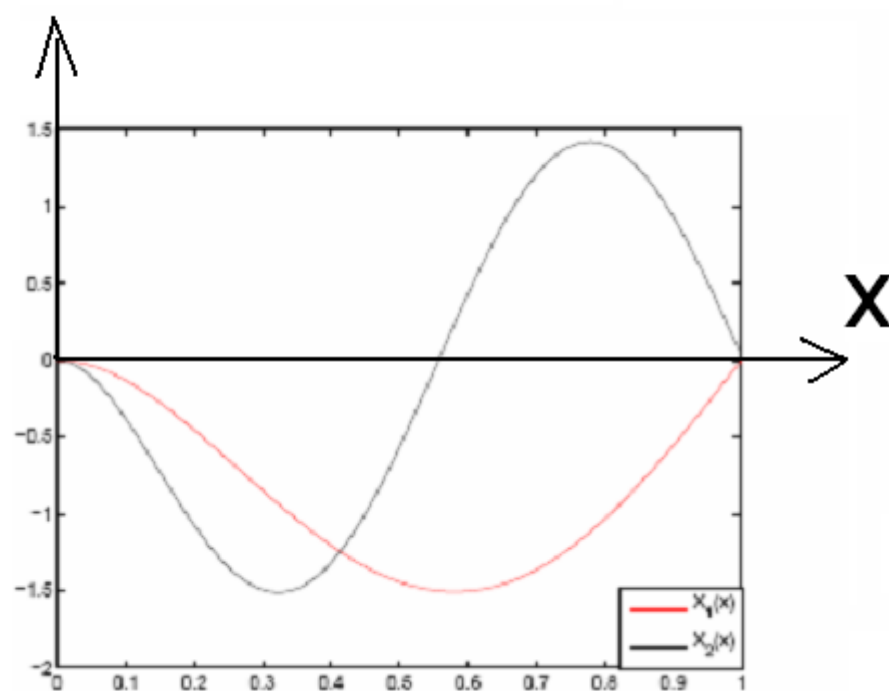
$$u(x,0) = \sum_n c_n X_n(x)$$

$$y(x,0) = \sum_n c_n X_n(x)$$



$X_1(x)$

$X_2(x)$



Final Monday, December 12, 2005,
Room: 141 DeBartolo,
Time: 1:45-3:45 PM.

8 problems

Please bring:

1. 3 cheat sheets
2. calculator
3. pencils
4. scratch paper

It is formatted like the midterms: eight problems
total; 15 minutes/problem.