Relativistic transformation of perpendicular velocity components from the constancy of the speed of light

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Mermin has derived the relativistic addition of the parallel components of velocity directly from the constancy of the speed of light. In this note the derivation is extended to the perpendicular components of the velocity.
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Mermin[1] has given a succinct and elementary derivation of the relativistic addition of the parallel component of the velocity using only the constancy of the speed of light, without resorting to Lorentz transformations. In this note I show that his derivation can be extended to the transformation of the perpendicular components of the velocity.

As in Ref. 1 we consider a train moving on a long straight track at constant speed $V_\parallel$. At a certain instant, a photon, which has speed c, and a massive particle, which has speed less than c, are both projected from one side the train, say the right side, in the direction perpendicular (according to someone standing on the train) to the direction of the train. When the photon reaches the left side of the train, it is immediately reflected back. On its way back to the right side, it encounters the massive particle at some point.

Because the point on the train where the photon and the massive particle meet is frame independent, the ratio r of the perpendicular components of the velocity of the massive particle and that of the photon must be an invariant. To a person standing on the ground, the parallel components of the velocities of both the massive particle and the photon are $V_\parallel$, the speed of the train. Because the speed of the photon is constant at c in any frame, the perpendicular component of its velocity according to the person standing on the ground is, by Pythagoras' theorem, $\sqrt{c^2-V_\parallel^2}$. We let $V_\perp$ be the perpendicular component of the velocity of the massive particle as observed by the person on the ground. The invariant ratio is

$$r = \frac{V_\perp}{\sqrt{c^2 - V_\parallel^2}}. \qquad (1)$$

Consider two cases for the speed of the train, $V_\parallel = u_\parallel$ and $V_\parallel = w_\parallel$. According to the person standing on the ground, the parallel components of the velocity of the massive particle are $u_\parallel$ and $w_\parallel$, and its perpendicular components are $u_\perp$ and $w_\perp$, respectively. The invariance of r gives

$$r = \frac{u_\perp}{\sqrt{c^2 - u_\parallel^2}} = \frac{w_\perp}{\sqrt{c^2 - w_\parallel^2}}, \qquad (2)$$

which implies that

$$w_\perp = u_\perp \sqrt{\frac{c^2 - w_\parallel^2}{c^2 - u_\parallel^2}}. \qquad (3)$$

All that is left to do is to express $w_\parallel$ in terms of the velocity difference v between $w_\parallel$ and $u_\parallel$ using the parallel velocity addition rule[1]

$$w_\parallel = \frac{u_\parallel + v}{1 + (u_\parallel v/c^2)}. \qquad (4)$$

If we substitute Eq. (4) into Eq. (3), we obtain the desired result[2]

$$w_\perp = \sqrt{1 - (v^2/c^2)}\frac{u_\perp}{1 + (u_\parallel v/c^2)}. \qquad (5)$$

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[1]N. David Mermin, "Relativistic addition of velocities directly from the constancy of the velocity of light," Am. J. Phys. **51**, 1130–1131 (1983). Also, see N. David Mermin, *It's About Time: Understanding Einstein's Relativity* (Princeton University Press, Princeton, 2005), Chap. 4, for a discussion of this derivation that is more accessible to general audiences.

[2]See, for example, David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Upper Saddle River, NJ, 1999), 3rd ed., p. 508; John R. Taylor, *Classical Mechanics* (University Science Books, Sausalito, CA, 2005), p. 616.