Energy and momentum transport in string waves

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Formulas are derived for the energy, momentum, and angular momentum transmitted by waves of arbitrary shape in an inextensible string and by pure transverse waves in a suitably extensible string. The analysis is based on Tait's procedure of viewing the wave from a moving frame of reference, and its application to the inextensible string, at least, is simple enough for use in elementary physics courses. The same method is used to find the elastic condition on an extensible string necessary for the propagation of mutually coherent shape and density waves. It is shown that the perfect transverse string waves referred to in textbooks can only be propagated in strings of the ideal Slinky spring type, and that they are incapable of carrying net momentum.

I. INTRODUCTION

Typical elementary textbooks introduce the subject of wave propagation with the picturesque example of a traveling deformation in a string. Many authors correctly point out that the string wave carries energy with it, and then proceed to calculate the rate of energy transfer in the limited context of a traveling sinusoid of small amplitude or of a more general, but still small, solution of the ordinary wave equation. In contrast, the possibility of momentum transport by the string wave is rarely mentioned, and there is a remarkable lack of elaboration of that matter at the elementary level. Even among the more advanced texts I know of only one[1] in which an analysis of the problem is attempted. Information is not quite so hard to find in the more practical area of momentum transfer by an acoustical wave in a fluid. French[2] has given some space to a qualitative discussion of that topic in one of his introductory texts, for instance. Any quantitative treatment inevitably gets postponed to works intended at least for the junior–senior level,[1,3] however, and nowadays few students—even physics majors—are likely to see it at all.

Foreseeing the introduction of the concepts of electromagnetic radiation pressure and the interrelation of the momentum, angular momentum, and energy carried by a photon or by a Schrödinger wave, a physics teacher might find it worthwhile to show that analogous properties belong to waves in general or to the string wave in particular. I propose the development in the following two sections of this paper as a suitable means to the latter end. Section II, which deals with the basic method of attack and its application to the plane-polarized wave, could be presented to students who have developed some physical insight but have not yet mastered college-level mathematical skills, since it makes no explicit use of the wave equation, nor even of formal calculus. Section III depends on a slightly more sophisticated knowledge of vector algebra and coordinate systems for the description of a more general deformation in a string and the matter of angular momentum transport. The model I have adopted for the string in both sections is not new but it is a little out of the ordinary, and as a result my formula for the rate of energy transfer differs significantly from the traditional one. I have attempted to resolve the differences in Sec. IV, and to point out some ambiguities in the standard treatment; a little calculus has had to be used for those purposes.

II. BASIC PROCEDURE AND ITS APPLICATION TO PLANE-POLARIZED STRING WAVES

An appropriate place to begin a survey of the dynamical properties of waves is the imaginative derivation, apparently originated by Tait,[4] of the propagation velocity of a string wave. The derivation is paraphrased in a number of modern texts,[5] but seldom, if ever, is it credited with the generality it seems to have in fact.[6] In substance, it examines the dynamics of curvilinear motion of a segment of the string as seen in a reference frame moving with the wave propagation velocity **c**. In that frame every segment is moving backward along the waveform with speed c, and is constrained to follow the local curvature of the waveform by the resultant of the tensile forces acting on its ends. When that resultant is identified with the centripetal force on the segment, the familiar relation

$$c = (T_0/\sigma_0)^{1/2} \tag{1}$$

follows, where T_0 is the tension and σ_0 the linear density. Since the orientation, length, and instantaneous radius of curvature do not appear in this exact result, the latter two quantities could be arbitrarily small, so that Eq. (1) carries the implied applicability to an arbitrary waveform. As for the properties of the string, it must be assumed to be perfectly flexible. In the simplest situation we can suppose it to be inextensible, with the tension T_0 supplied by stretched springs or suspended weights at the remote ends; those devices will be in a state of elevated potential energy after the deformation has been introduced. A string with longitudinal elasticity may be used, but in that event the corresponding excess potential energy must be distributed uniformly along the entire length of the string, including that additional length needed to construct the waveform. In neither case is the potential energy of deformation propagated with the wave in its entirety. By keeping T_0 and σ_0 uniform along the deformed string, we automatically exclude any tangential acceleration of the string segment in the moving frame of reference and provide for a waveform that does not change in time. An alternative model will be discussed in Sec. IV.

It will be convenient to formulate the problem of energy and momentum transport in a coordinate system whose xy plane is perpendicular to, and whose z axis coincides with, the undeformed length of the string. For the time being we can examine the case in which the

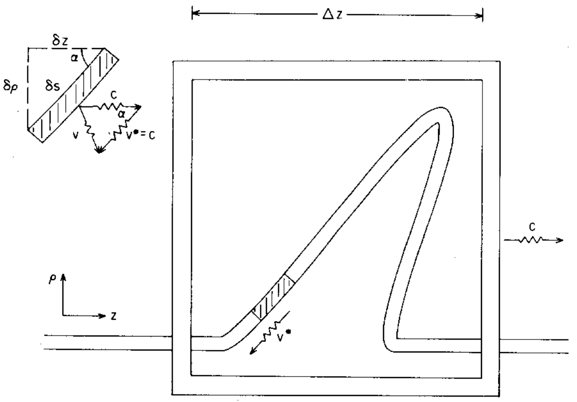

Fig. 1. A possible waveform in an inextensible string, as viewed in a reference frame moving in the axial (z) direction with the propagation speed c. An enlargement of a typical string segment at an instant during the passage of the wave is shown at upper left. Note that there is no such segment whose lab-frame velocity $\mathbf{v}$ has a component in the negative z direction; all momentum is carried forward.

propagating deformation is confined to a plane. To avoid the necessity of reformulating the results subsequently in the more general three-dimensional problem, we shall use cylindrical coordinates, with the plane of polarization at constant angle φ with the xz plane and the instantaneous position of any particle in the string described by its coordinates ρ and z.

Figure 1 represents a possible deformation traveling in the positive z direction. (To emphasize the arbitrariness in choice of the waveform, I have deliberately shown a multivalued function.) We inspect a representative short segment of length δs. Instantaneously it is oriented at angle α relative to the z axis, has projected lengths δz and $\delta\rho$ along and perpendicular to that axis, and is moving with velocity $\mathbf{v}$ in the laboratory frame of reference.

To evaluate $\mathbf{v}$ we resort again to Tait's moving frame of reference and find that the vector in question is a composite of the forward velocity $c\hat{\mathbf{z}}$ of that frame and the velocity $\mathbf{v}^*$, also of magnitude c, of the segment as seen in the frame. Referring to the diagram in Fig. 1 and making use of the law of cosines, we obtain the relationship

$$v^2 = 2c^2(1-\cos\alpha) = 2c^2(1-\delta z/\delta s), \qquad (2)$$

from which the instantaneous kinetic energy of the segment δs is found to be

$$\delta K = \tfrac{1}{2}\sigma_0 \delta s v^2 = \sigma_0 c^2(\delta s - \delta z). \qquad (3)$$

While we are about it, we can also evaluate the segment's momentum components. Using the components of the lab-frame velocity described in Fig. 1, we obtain

$$\delta p_z = \sigma_0 \delta s v_z = \sigma_0 \delta s c(1-\cos\alpha) = \delta K/c \qquad (4)$$

and

$$\delta p_\rho = \sigma_0 \delta s v_\rho = -\sigma_0 \delta s c \sin\alpha = -\sigma_0 c \delta\rho. \qquad (5)$$

The entire pulse, or some section of interest, involves a string length Δs whose forward extremity is separated from its rear by an axial interval Δz. The quantities $\Delta s - \Delta z$ and

$$\Delta m = \sigma_0(\Delta s - \Delta z) \qquad (6)$$

would represent the excess length and mass of string that have been accumulated into the interval Δz in order to construct the deformation. Thus a simple summation can be performed on segment kinetic energies as described in Eq. (3), and we find that the total energy (entirely kinetic) carried by the pulse is

$$\Delta E = \Delta m c^2. \qquad (7)$$

The corresponding longitudinal and transverse momenta borne by the pulse are

$$\Delta p_z = \Delta E/c = \Delta m c \qquad (8)$$

and

$$\Delta p_\rho = -\sigma_0 c \Delta_\rho = 0. \qquad (9)$$

The analogy to the electromagnetic energy–momentum relations is obvious in Eqs. (7) and (8).

III. STRING WAVES INVOLVING THREE DIMENSIONS

A slightly more elaborate treatment is needed for the discussion of a more general deformation of a string, a possible instance of which is shown in Fig. 2. Using all three cylindrical coordinates, we can characterize the instantaneous length and orientation of the typical short string segment by the vector

$$\delta\mathbf{s} = \delta\rho\hat{\rho} + \rho\delta\varphi\hat{\varphi} + \delta z\hat{\mathbf{z}}, \qquad (10)$$

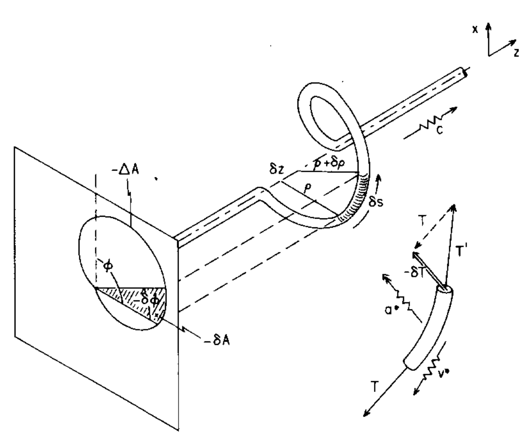

Fig. 2. A three-dimensional deformation propagating along a string in the positive z direction, and its projection on a transverse plane. In the enlarged version of the segment δs shown at lower right, the forces $\mathbf{T}$ and $\mathbf{T}'$ are shown with different magnitudes, as could be the case for an extensible string. For an inextensible string the forces would be equal and their resultant $-\delta\mathbf{T}$ would be directed toward the segment's center of curvature.

where $\hat{\rho}$, $\hat{\varphi}$, and $\hat{\mathbf{z}}$ are unit vectors and φ is now a variable angle. In the frame of reference of the moving waveform, the segment in question has a velocity $-c\delta\mathbf{s}/\delta s$. Its velocity in the laboratory is therefore

$$\mathbf{v} = c(\hat{\mathbf{z}} - \delta\mathbf{s}/\delta s). \tag{11}$$

With the help of Eq. (10), the square and the ρ and z components of $\mathbf{v}$ can be evaluated and combined with the segment's mass $\sigma_0\delta s$ to produce the same set of expressions, Eqs. (3)–(9), obtained in Sec. II for the plane-polarized disturbance. In addition we can now compute the angular momentum carried by one or all of the string segments associated with the pulse. The component of particular interest is the one directed along the z axis, since we might anticipate an extension of the electromagnetic analogy. The moving string segment δs makes the contribution

$$\delta L_z = \sigma_0 \delta s \rho v_\varphi$$

with respect to a point on the axis, and Eqs. (10) and (11) can be used to rewrite this in the form

$$\delta L_z = -\sigma_0 c \rho^2 \delta\varphi. \tag{12}$$

As the segment length δs under consideration is reduced toward zero, the quantity $\rho^2\delta\varphi$ approaches twice the projected area on the xy plane of the triangle formed by $\delta\mathbf{s}$ and the position vectors to its ends:

$$\lim_{\delta s \to 0} \delta L_z = -2\sigma_0 c \delta A_z. \tag{13}$$

Here, the contribution δA_z to the projected area has been defined as positive if it is generated by a clockwise rotation of the position vector as seen from the wave source, that is, a positive $\delta\varphi$. Thus the total z component of angular momentum conveyed by the pulse is exactly

$$\Delta L_z = -2\sigma_0 c \Delta A_z, \tag{14}$$

where ΔA_z is the accumulated area of the pulse's projection on the xy plane, as shown in Fig. 2.

An interesting special case is that of a helical (circularly polarized) pulse of constant radius R and constant pitch $2\pi c/\omega$, in which ω denotes the angular frequency at which turns of the helix pass a point in the laboratory. For a helix of N cycles, the angular momentum prescribed by Eq. (14) is

$$\Delta L_z = \pm 2\pi N \sigma_0 c R^2, \tag{15}$$

where the signs represent the alternate senses of the circular polarization. If N is large enough to warrant the neglect of end contributions, the pulse length and string length for this deformation are

$$\Delta z = 2\pi N c/\omega \tag{16}$$

and

$$\Delta s = 2\pi N[(c/\omega)^2 + R^2]^{1/2}. \tag{17}$$

From Eqs. (6), (7), and (15), it follows that the energy carried by such a pulse is

$$\begin{aligned}\Delta E &= 2\pi N \sigma_0 c^3 \omega^{-1}[(1+\omega^2R^2/c^2)^{1/2} - 1] \\ &= \omega(\omega R/c)^{-2}[(1+\omega^2R^2/c^2)^{1/2} - 1]\,|\Delta L_z|. \end{aligned} \tag{18}$$

This relationship is not as simple as its electromagnetic analog, for which the energy is the product of ω and the angular momentum. However, in the extreme situation where the radius of the helical deformation is much smaller than its pitch, the limiting value of Eq. (18) is

$$\lim_{\omega R/c \to 0} \Delta E = |\Delta L_z|\,\omega/2. \tag{19}$$

IV. COMPARISON OF UNIFORM-DENSITY STRING WITH THE CONVENTIONAL MODEL

The model string with uniform tension and linear density, as I have used it in the previous two sections, is fundamentally different from the variable-tension, variable-density model commonly analyzed in textbooks as a carrier of wave energy. It is not unusual to find both models referred to in the same context, however, as though there were no distinction between them. Hereafter, I refer to the two models as R (for "rope-like") and S (for "spring-like").

It should be evident that the type-S string can, and typically will, transmit two kinds of wave, one in the tension or linear density and the other of the kind already described in connection with the type-R string. I shall call these "density" and "shape" modes of propagation. The density mode is essentially similar to the longitudinal wave in an elastic rod, and in that connection often gets treated separately. The shape wave, on the other hand, is considered by itself only in those textbooks that make use of the Tait derivation of the propagation speed, and then only in that limited respect. Otherwise it is traditionally discussed in a way that associates it implicitly with a density wave. It is not hard to show that the string must have a particular elastic property in order for the two modes to be propagated with the same speed; even then the modes are not coupled in a perfectly flexible string, so that any connection there might be between the two wave functions must be imposed by the wave source, and is not a property of the medium. I shall derive the necessary elastic condition here in a simple extension of the Tait-type analysis used previously; a more formal derivation and the development of a wave equation will be presented in the Appendix.

When the shape and density modes share a common speed c, the flow of string will be steady at any point in a Tait frame of reference moving with the waveform. The local linear density σ and flow speed v^* in such a frame can be related by continuity of mass flow:

$$\sigma v^* = \sigma_0 c, \tag{20}$$

in which σ_0 and c are the density and observed speed at an axial point where the string tension is T_0. With reference to the diagram in Fig. 2, the net force, $-\delta\mathbf{T} = \mathbf{T} + \mathbf{T}'$, on a short string segment of length δs may be equated to the product of the segment's mass and its observed acceleration in the Tait frame:

$$-\delta\mathbf{T} = \sigma\delta s\mathbf{a}^*.$$

There being no explicit time dependence of $\mathbf{T}$ or $\mathbf{v}^*$ in this reference frame, the last equation can be rewritten in the form

$$\delta s\partial\mathbf{T}/\partial s = \sigma\delta s v^* \partial\mathbf{v}^*/\partial s,$$

and using the constancy of σv^* expressed in Eq. (20), we obtain

$$\partial(\mathbf{T} - \sigma v^* \mathbf{v}^*)/\partial s = 0.$$

The vector quantity in parentheses must be conserved in the flow, and in fact must be zero everywhere, since $\mathbf{T}$ and $\mathbf{v}^*$ are both tangent to the string at any point. Thus, at every point we must have

$$v^* = (T/\sigma)^{1/2}, \quad (21)$$

and it follows from Eq. (20) that the elastic property the string must have if its shape and density waves are to be propagated at equal speeds is described by

$$T\sigma = T_0\sigma_0. \quad (22)$$

This is just the property of an ideal spring with zero relaxed length, a condition approximated by the Slinky spring that is often used for wave demonstrations. Some other interdependence of tension and density, for example, the negligible Young's modulus suggested by Elmore and Heald,[1] must result either in a notable difference in propagation speeds of the two modes, or in no wave at all. At the other extreme from the very weak spring just alluded to is the type-R string, whose Young's modulus is so far in excess of the tensile stress that any local variation in tension is immediately propagated away from the shape wave of interest in Secs. II and III, leaving Eq. (22) trivially satisfied by separately constant values of T and σ.

Further reference to Fig. 2 and to Eq. (20) will let us evaluate the kinetic energy of the string segment δs in the laboratory frame:

$$\delta K = \tfrac{1}{2}\sigma\delta s v^2 = \tfrac{1}{2}\sigma\delta s c^2[\hat{\mathbf{z}} - (\sigma_0/\sigma)(\delta\mathbf{s}/\delta s)]^2$$
$$= \tfrac{1}{2}\sigma c^2[(\sigma_0/\sigma)^2 + 1]\delta s - \sigma_0 c^2\delta z. \quad (23)$$

To this we must add the potential energy of density variation relative to the standard (undeformed) density σ_0, namely,

$$\delta U = \tfrac{1}{2} T\delta s(1 - \sigma^2/\sigma_0{}^2).$$

This can be put in terms compatible with δK by means of Eqs. (20)–(22), so that we get

$$\delta U = \tfrac{1}{2}\sigma c^2[(\sigma_0/\sigma)^2 - 1]\delta s, \quad (24)$$

and the total energy of a segment of the type-S string can be written

$$\delta E = \delta K + \delta U$$
$$= \sigma_0 c^2[(\sigma_0/\sigma)\delta s - \delta z]. \quad (25)$$

The transverse and axial components of the segment's linear momentum are

$$\delta p_\rho = -\sigma_0 c\delta\rho \quad (26)$$

and

$$\delta p_z = \sigma_0 c[(\sigma/\sigma_0)\delta s - \delta z]. \quad (27)$$

For $\sigma = \sigma_0$, Eqs. (25) and (27) reduce to their type-R counterparts, Eqs. (3) and (5). However, in the general case of an arbitrary deformation of a type-S string, the ratio of a segment's energy to its forward momentum is quite definitely not c, but depends on the particular admixture of shape and density waves introduced by the wave source.

The strong emphasis on the transverseness of string particle motion, as found in popular elementary texts, may mislead the student to think of such motion as a property of the medium rather than of the source, which it really is. Once the source has imparted a transverse motion, any string segment subsequently involved in the wave propagation will have its density determined by the relationship

$$\sigma\delta s = \sigma_0\delta z. \quad (28)$$

In that situation, according to Eq. (27), no segment makes any contribution to forward momentum transport and, from Eq. (26), a complete pulse or a period of a continuous wave carries no net transverse momentum either. In short, the typical textbook string wave is not a carrier of momentum in the nonrelativistic sense. Moreover, contrary to common usage, there is no limitation on the wave function and its derivatives, for pure transverse string waves, other than single-valuedness. (The form shown in Fig. 1 is precluded, for example.) Amplitudes are restricted only by the elastic limit of the medium.

When the transverse wave condition in Eq. (28) is applied to Eq. (25), we obtain the energy resident in a segment having instantaneous axial projection δz. It is

$$\delta E = \sigma_0 c^2[(\sigma_0/\sigma)^2 - 1]\delta z$$
$$= T_0(\partial\rho/\partial z)^2\delta z, \quad (29)$$

which is the usual textbook formula. This energy is divided equally between kinetic and potential forms at all times. For small-amplitude motion it is about twice the energy resident in a comparable segment of type-R string, in which the potential energy of deformation is a static property. The coresidence of equal kinetic and potential energies in every string segment is no more a property of the type-S string than is the absence of momentum, however; like the latter, it is a consequence of the initial conditions that impart transverse motion to the string's particles. It should be possible to initiate either a pure shape wave or a pure density wave in such a string.

To complete the record of dynamical properties of the type-S string wave, we should find the angular momentum it is capable of carrying. The procedure in Sec. III is still appropriate, but for the segment speed in the Tait frame of reference we must use v^* from Eq. (20). The general form for the resulting angular momentum carried by a string segment will again be that of Eqs. (12) and (13), but with σ_0 replaced by the segment's linear density σ. For the special case of a transverse helical pulse, the combination of this angular momentum formula with the condition (28) and the first form of the energy expression (29) yields the relationship

$$\delta E = |\delta L_z| \omega(\sigma_0/\sigma). \tag{30}$$

The likeness to the comparable electromagnetic formula improves with decreasing radius of the helix.

V. SUMMARY

Although it is regularly displayed in teaching literature only as a trick device for deriving the wave propagation speed in a string, the Tait procedure of viewing a wave from a frame of reference in which it appears static has a more general usefulness. As I have indicated in this paper, it is also helpful in describing the particular stress–strain property an extensible string must have in order to convey an unchanging waveform and in calculating the densities of momentum and energy carried by string waves generally.

With regard to the latter dynamical transport, I have shown that a deformation of any shape in a flexible, inextensible (type-R) string carries energy and axial linear momentum, both positive in sign, in the definite ratio c, the speed of wave propagation. The axial angular momentum turns out to be proportional to the area enclosed by a projection of the wave shape on a transverse plane.

Deformations in an extensible string of the kind approximated by the extended Slinky spring (type-S string) behave like an arbitrary mixture of the shape wave in the type-R string and a density wave, both propagating at the same speed; energy and momentum may be of the same or different algebraic sign and they are not simply related. For the special case in which such a string, of normal linear density σ_0, is excited by a purely transverse sinusoidal source of angular frequency ω and amplitude ρ_{max}, each wavelength conveys energy in the well-known amount $\sigma_0\omega^2\rho_{max}^2/2$, as can be verified readily from Eq. (29). This most popular of all textbook wave examples carries no net momentum. The description of angular momentum transport in this case in similar to that for the type-R string.[7]

Our familiarity with electromagnetic radiation may condition us to expect that forward momentum, in some such amount as energy/propagation speed, is the inevitable companion of any kind of wave. Not long ago Vigoureux[8] revived an early speculation by Poynting[9] to that effect. Poynting's immediate concern had been the radiation pressure which was known to be exerted on surfaces irradiated by either light or sound. It was not yet realized at the time of his writing, 1905, that the momenta carried by both types of wave could be associated with mass transfer. In the optical case the mass in question was soon to be identified by the special relativity theory. In the acoustical case, as Rayleigh[10] would show later in 1905, the mass—hence, momentum—transport depends on the different speeds at which regions of condensation and rarefaction are propagated, and hence on the equation of state of the medium.[11] Rayleigh went on to show that, in fluids of certain hypothetical pressure–density behavior, a wave might carry zero or even negative momentum. Notwithstanding the unnatural properties he had to invoke for such fluids, he can be credited with indicating that forward momentum, although common, is not a necessary attribute of waves as such; rather its presence or absence depends on the particular type of wave and the properties of the medium that carries it.

As in waves of other kinds, the transport of forward momentum in string waves implies that mass is also being transferred and that the source must somehow serve as a dispenser for that mass. Such mass flow is evident in progressive waves propagated along an inextensible type-R string, and it finds everyday use by nonphysicists who want to displace ropes, carpets, or bedsheets forward along their lengths or in their planes.[12] It is almost as evident that a true transverse wave in the type-S string, as usually described or implied in textbooks, involves no mass transfer at all. The source of this kind of wave, being fastened to the string and constrained to move only in a transverse plane, is in no sense a mass dispenser and so is not a source of momentum.

APPENDIX

The wave equation for a string can be derived and the elastic property necessary for the propagation of an undistorted wave can be determined as follows.[13] (Note that the Lagrangian representation of the motion of a general string particle is being used, rather than the more conventional Eulerian description of a displacement or velocity field.)

Let the extremities of a short string segment of fixed mass m be instantaneously located at positions $\mathbf{r}'$ and $\mathbf{r}' + \delta\mathbf{r}'$, relative to their original positions $z\hat{\mathbf{z}}$ and $(z + \delta z)\hat{\mathbf{z}}$. Calling this directed segment $\delta\mathbf{s}'$, we can describe it by

$$\delta\mathbf{s}' = \delta z\hat{\mathbf{z}} + \delta\mathbf{r}' = (\hat{\mathbf{z}} + \partial\mathbf{r}'/\partial z)\delta z. \tag{A. 1}$$

The original and instantaneously deformed lengths of the segment δz and $\delta s'$ are related by the conservation of mass:

$$\delta m = \sigma\delta s' = \sigma_0\delta z. \tag{A. 2}$$

The forces exerted on the segment by adjoining string are $\mathbf{T}(z)$ and $-\mathbf{T}(z+\delta z)$, where $\mathbf{T}$ is given by

$$\mathbf{T} = -T\delta\mathbf{s}'/\delta s' = -(T\sigma/\sigma_0)\delta\mathbf{s}'/\delta z, \qquad (\text{A.}\,3)$$

so that the net force is

$$\begin{aligned}\mathbf{T}(z) - \mathbf{T}(z+\delta z) &= -(\partial\mathbf{T}/\partial z)\delta z \\ &= (T\sigma/\sigma_0)(\partial^2\mathbf{r}'/\partial z^2)\delta z \\ &\quad + (\delta\mathbf{s}'/\sigma_0)\partial(T\sigma)/\partial z. \qquad (\text{A.}\,4)\end{aligned}$$

From the preceding equations and Newton's second law we obtain the local acceleration

$$\begin{aligned}\partial^2\mathbf{r}'/\partial t^2 &= (T\sigma/\sigma_0{}^2)\partial^2\mathbf{r}'/\partial z^2 \\ &\quad + \sigma_0{}^{-2}(\hat{\mathbf{z}} + \partial\mathbf{r}'/\partial z)\partial(T\sigma)/\partial z. \qquad (\text{A.}\,5)\end{aligned}$$

The second term on the right vanishes under the condition that $T\sigma$ has the same value throughout the string, including the undeformed portions, where T and σ are T_0 and σ_0. Thus, for

$$T\sigma = T_0\sigma_0, \qquad (\text{A.}\,6)$$

Eq. (A.5) reduces to the ordinary wave equation

$$\partial^2\mathbf{r}'/\partial t^2 = (T_0/\sigma_0)\partial^2\mathbf{r}'/\partial z^2, \qquad (\text{A.}\,7)$$

which yields the same results as were obtained by a Tait analysis in Sec. IV.

[1]W. C. Elmore and M. A. Heald, *Physics of Waves* (McGraw-Hill, New York, 1969), Chap. 1.

[2]A. P. French, *Vibrations and Waves* (Norton, New York, 1971), p. 243.

[3]R. B. Lindsay, *Mechanical Radiation* (McGraw-Hill, New York, 1960), Chap. 9; I. Tolstoy, *Wave Propagation* (McGraw-Hill, New York, 1973), Chap. 10. The latter text introduces the "Stokes drift," a mass-transport effect known in the field of surface gravity-wave propagation and particularly relevant to the present discussion.

[4]P. G. Tait, *Encyclopedia Brittanica* (Scribner's, New York, 1883), 9th ed., Vol. XV, p. 741.

[5]D. Halliday and R. Resnick, *Fundamentals of Physics* (Wiley, New York, 1970), p. 307; R. T. Weidner and R. L. Sells, *Elementary Classical Physics* (Allyn and Bacon, Boston, 1973), 2nd ed., p. 302; C. A. Coulson, *Waves* (Oliver and Boyd, Edinburgh, 1955), p. 40; P. M. Morse, *Vibration and Sound* (McGraw-Hill, New York, 1948), 2nd ed., p. 72.

[6]P. M. Morse [*Vibration and Sound* (McGraw-Hill, New York, 1948)] writes, "Now *if* the tube is not bent too sharply anywhere . . . the tension in the string will everywhere be T dynes." Two paragraphs later he adds, ". . . as long as the string has a uniform density ϵ g per cm, and as long as the displacement of the string is not too great, then a wave will travel along the string with a velocity $c = (T/\epsilon)^{1/2}$ regardless of the form of the wave." Tait himself put no such qualifications on the permissible deformations. He writes (cf. Ref. 4) regarding his proof, "It is to be observed that there is no necessity for limiting the proposition above to a plane curve, though we have treated the question as if it were such. The demonstration applies even to a knot [sic] of any form."

[7]In discussing the motion of the type-S string I deliberately avoided the complicating possibility of yet another, torsional, deformation accompanying the density wave along the string's contour.

[8]P. Vigoureux, Contemp. Phys. **7**, 440 (1966). [I am indebted to one of the referees of the present paper for bringing this reference to my attention.]

[9]J. H. Poynting, Philos. Mag. **9**, 393 (1905).

[10]Lord Rayleigh, Philos. Mag. **10**, 364 (1905).

[11]It can be shown by a Tait analysis that the ratio of momentum to energy in the acoustic wave is $(\Pi_2/\Pi_1 + 2)/4c_0$, where Π_1 and Π_2 are the first and second adiabatic derivatives of pressure with respect to condensation and c_0 is the propagation speed, all evaluated at zero condensation. In an ideal gas the ratio has the value $(\gamma + 1)/4c_0$, in agreement with Rayleigh's result. The only modern text in which I have found direct reference to a dependence of momentum transport on the equation of state is that of French (cf. Ref. 2).

[12]In such instances the media are not usually under tension and the typical restoring forces responsible for wave propagation are gravitational. The mass transfer is similar to that in the type-R string.

[13]See J. B. Keller, Am. J. Phys. **27**, 584 (1959) for another derivation in a similar spirit.